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STUDY OF CRITICAL PARAMETERS FOR REACTOR SYSTEMS (COLLECTION OF ARTICLES)

By

Various Authors

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STUDY OF CRITICAL PARAMETERS FOR REACTOR SYSTEMS
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by Yu. A. Romanov

Kinetic Equation, Solution of Milne's Problem and Determination of Albedo of Semi-infinite Medium

The kinetic equation for the single-velocity plane problem for an isotropic scattering indicatrix is, as is well known,

$$\mu \frac{\partial \psi}{\partial z} + \psi = \frac{p}{2} \int_{-1}^{+1} \psi \, d\mu, \tag{1}$$

in which $\psi(z, \mu)$ is the distribution function of the number of particles;

μ is the cosine of the angle between the direction of flight of a particle and the Z axis;

p is the parameter describing the properties of a medium and is equal to the ratio of the scattering cross section and the total cross-section ($p < 1$).

The total path length of the particle is taken as unity.

In an unbounded medium Eq. (1) has the exact solution

$$\psi(z, \mu) = \frac{p}{2} \left[C_1 \frac{e^{kz}}{1 + k\mu} + C_2 \frac{e^{-kz}}{1 - k\mu} \right],$$

$$\psi_0(z) = \int_{-1}^{+1} \psi(z, \mu) \, d\mu = C_1 e^{kz} + C_2 e^{-kz}, \tag{2}$$

in which $\psi_0(x)$ is the particle density;

C_1 and C_2 are arbitrary constants;

k is the coefficient which is found from the solution of the transcendental equation

$$p\frac{\operatorname{Arth} k}{k} = 1. \tag{3}$$

In a breeding medium, at $p > 1$, the roots of Eq. (3) are purely imaginary, while the solutions of Eq. (2) are periodic.

The solutions of Eq. (2) satisfy the diffusion equation

$$D_0 \Delta\psi_0 + (p-1)\psi_0 = 0, \tag{4}$$

provided the diffusion factor in it is assumed equal to

$$D_0 = \frac{1-p}{\mu^2} \tag{5}$$

(both the velocity of the particle and its path length are equal to unity). Solution (2) describes the behavior of the particle density some way from the interface and from the sources with fair accuracy. From now on the particle density expressed by Eq. (2) will be known as the asymptotic density, while the

diffusion coefficient D_0 is known as the asymptotic diffusion coefficient. Table I gives the coefficients k and D_0 for different parameters p over the range $0.25 < p < 2.0$*.

The solution of Eq. (1) in a semi-infinite medium (Milno's problem) can be found by the Wiener and Hopf method /1/ and is given in /2-4/ for the particular case of a non-absorptive medium ($p = 1$). Ref. /5/ gives a solution of the Milne problem at $p < 1$ by the variation method. The particle density function ψ_0 in Milno's problem can be represented as

$$\psi_0 = C_1 e^{kz} + C_2 e^{-kz} + O(e^{-z}) = n(z) + f(z). \tag{6}$$

The asymptotic part of the solution ψ_0 (from now on called the asymptotic density), as can be seen from the numerical data in /5/, differs from the true density $\psi_0(0)$ on the boundary ($z=0$) by approximately 20%, and at the distance of the semipath from the boundary ($z=0.5$) by less than 3%. The asymptotic density $n(z)$ in Milne's problem is equal to** (see next page)

$$n(z) = \psi_0(0)\sqrt{\frac{2(1-k^2)}{k^2-1+p}}\operatorname{sh} k(z+z_0). \tag{7}$$

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* For $p > 1$ k can be found with good accuracy from the following formula $k - \frac{3}{k^2} = \frac{1}{p(p-1)} + \frac{1}{3} - \frac{1}{3p^2}$

Table 1

Values of coefficients k, D_0, γ

p	k	D_0	γ	p	k	D_0	γ
0,25	0,99932	0,75102	0,1171	0,68	0,84707	0,44600	0,7070
0,26	0,99908	0,74136	0,1332	0,69	0,83804	0,44148	0,7176
0,27	0,99878	0,73179	0,1492	0,70	0,82862	0,43701	0,7281
0,28	0,99840	0,72231	0,1651	0,71	0,81884	0,43260	0,7384
0,29	0,99794	0,71294	0,1810	0,72	0,80865	0,42828	0,7486
				0,73	0,79804	0,42402	0,7587
0,30	0,99741	0,70364	0,1969	0,74	0,78700	0,41981	0,7687
0,31	0,99678	0,69447	0,2127	0,75	0,77551	0,41566	0,7786
0,32	0,99605	0,68540	0,2285	0,76	0,76354	0,41157	0,7885
0,33	0,99521	0,67647	0,2442	0,77	0,75108	0,40760	0,7982
0,34	0,99424	0,66767	0,2599	0,78	0,73808	0,40375	0,8079
0,35	0,99316	0,65898	0,2755	0,79	0,72454	0,39998	0,8174
0,36	0,99195	0,65043	0,2910				
0,37	0,99059	0,64202	0,3064	0,80	0,71041	0,39628	0,8268
0,38	0,98909	0,63375	0,3217	0,81	0,69565	0,39262	0,8362
0,39	0,98743	0,62563	0,3369	0,82	0,68024	0,38900	0,8455
				0,83	0,66411	0,38542	0,8547
0,40	0,98562	0,61763	0,3520	0,84	0,64724	0,38189	0,8638
0,41	0,98364	0,60979	0,3669	0,85	0,62950	0,37842	0,8728
0,42	0,98150	0,60207	0,3816	0,86	0,61087	0,37501	0,8818
0,43	0,97917	0,59451	0,3962	0,87	0,59127	0,37166	0,8907
0,44	0,97667	0,58707	0,4107	0,88	0,57059	0,36839	0,8995
0,45	0,97397	0,57979	0,4251	0,89	0,54868	0,36520	0,9083
0,46	0,97109	0,57263	0,4394				
0,47	0,96800	0,56562	0,4535	0,90	0,52543	0,36206	0,9170
0,48	0,96473	0,55872	0,4673	0,91	0,50061	0,35897	0,9256
				0,92	0,47397	0,35593	0,9341
0,49	0,96122	0,55198	0,4808	0,93	0,44524	0,35293	0,9426
0,50	0,95750	0,54537	0,4940	0,94	0,41394	0,34998	0,9510
0,51	0,95357	0,53888	0,5071	0,95	0,37948	0,34708	0,9593
0,52	0,94941	0,53252	0,5201	0,96	0,34081	0,34423	0,9676
0,53	0,94502	0,52628	0,5329	0,97	0,29625	0,34143	0,9758
0,54	0,94040	0,52016	0,5455	0,98	0,24305	0,33868	0,9839
0,55	0,93553	0,51416	0,5579	0,99	0,17227	0,33598	0,9920
0,56	0,93041	0,50828	0,5702				
0,57	0,92504	0,50251	0,5824	1,00	0,0000	0,33333	1,0000
0,58	0,91940	0,49686	0,5945	1,01	0,17383	0,33074	1,0080
0,59	0,91350	0,49132	0,6064	1,02	0,24686	0,32819	1,0160
				1,03	0,30356	0,32566	1,0238
0,60	0,90733	0,48588	0,6180	1,04	0,35188	0,32315	1,0316
0,61	0,90087	0,48055	0,6296	1,05	0,39497	0,32067	1,0393
0,62	0,89411	0,47534	0,6411	1,06	0,43431	0,31822	1,0470
0,63	0,88708	0,47019	0,6524	1,07	0,47092	0,31580	1,0546
0,64	0,87973	0,46516	0,6636	1,08	0,50585	0,31341	1,0622
0,65	0,87206	0,46023	0,6746	1,09	0,53802	0,31105	1,0698
0,66	0,86407	0,45539	0,6855				
0,67	0,85575	0,45063	0,6963	1,10	0,56926	0,30872	1,0773

p	k	D_0	τ	p	k	D_0	τ
1,11	0,59926	0,30641	1,0848	1,56	1,56216	0,22948	1,3813
1,12	0,62822	0,30113	1,0722	1,57	1,59046	0,22820	1,3872
1,13	0,65626	0,30189	1,0096	1,58	1,59869	0,22693	1,3931
1,14	0,68351	0,29968	1,1069	1,59	1,61687	0,22568	1,3989
1,15	0,71008	0,29750	1,1141				
1,16	0,73597	0,29535	1,1213	1,60	1,63501	0,22444	1,4047
1,17	0,76135	0,29323	1,1285	1,61	1,65309	0,22322	1,4105
1,18	0,78618	0,29114	1,1356	1,62	1,67112	0,22201	1,4163
1,19	0,81058	0,28908	1,1427	1,63	1,68910	0,22081	1,4221
				1,64	1,70704	0,21963	1,4279
1,20	0,83454	0,28705	1,1498	1,65	1,72493	0,21846	1,4336
1,21	0,85810	0,28505	1,1568	1,66	1,74279	0,21730	1,4393
1,22	0,88133	0,28309	1,1638	1,67	1,76059	0,21615	1,4450
1,23	0,90423	0,28116	1,1708	1,68	1,77835	0,21502	1,4506
1,24	0,92682	0,27925	1,1777	1,69	1,79609	0,21389	1,4562
1,25	0,94913	0,27738	1,1846				
1,26	0,97118	0,27555	1,1914	1,70	1,81378	0,21278	1,4618
1,27	0,99299	0,27375	1,1982	1,71	1,83143	0,21168	1,4674
1,28	1,01455	0,27197	1,2049	1,72	1,84905	0,21059	1,4730
1,29	1,03593	0,27021	1,2116	1,73	1,86663	0,20951	1,4785
				1,74	1,88419	0,20844	1,4840
1,30	1,05707	0,26846	1,2183	1,75	1,90170	0,20738	1,4895
1,31	1,07805	0,26672	1,2250	1,76	1,91920	0,20633	1,4950
1,32	1,09884	0,26501	1,2316	1,77	1,93664	0,20530	1,5005
1,33	1,11947	0,26332	1,2382	1,78	1,95407	0,20427	1,5059
1,34	1,13994	0,26165	1,2448	1,79	1,97146	0,20326	1,5113
1,35	1,16024	0,26000	1,2513				
1,36	1,18042	0,25836	1,2578	1,80	1,98882	0,20225	1,5167
1,37	1,20046	0,25675	1,2642	1,81	2,00617	0,20126	1,5221
1,38	1,22036	0,25516	1,2706	1,82	2,02348	0,20027	1,5275
1,39	1,24014	0,25358	1,2770	1,83	2,04076	0,19929	1,5329
				1,84	2,05802	0,19833	1,5382
1,40	1,25982	0,25202	1,2834	1,85	2,07525	0,19737	1,5435
1,41	1,27937	0,25049	1,2897	1,86	2,09246	0,19642	1,5488
1,42	1,29882	0,24897	1,2960	1,87	2,10964	0,19548	1,5541
1,43	1,31817	0,24747	1,3023	1,88	2,12681	0,19455	1,5594
1,44	1,33742	0,24599	1,3085	1,89	2,14396	0,19362	1,5646
1,45	1,35658	0,24452	1,3147				
1,46	1,37565	0,24308	1,3209	1,90	2,16106	0,19271	1,5698
1,47	1,39462	0,24165	1,3271	1,91	2,17815	0,19181	1,5750
1,48	1,41353	0,24023	1,3332	1,92	2,19524	0,19091	1,5802
1,49	1,43236	0,23883	1,3393	1,93	2,21228	0,19002	1,5854
				1,94	2,22932	0,18914	1,5906
1,50	1,45110	0,23745	1,3454	1,95	2,24633	0,18827	1,5957
1,51	1,46978	0,23608	1,3514	1,96	2,26333	0,18740	1,6008
1,52	1,48838	0,23473	1,3574	1,97	2,28031	0,18655	1,6059
1,53	1,50693	0,23339	1,3634	1,98	2,29725	0,18570	1,6110
1,54	1,52539	0,23208	1,3694	1,99	2,31422	0,18485	1,6161
1,55	1,54381	0,23077	1,3754	2,00	2,33112	0,18402	1,6212

in which z_0 is the extrapolated length, for which an expression is given below. The particle flux on the boundary in this case is

$$J(0) = \int_{-1}^{+1} \mu\psi(0, \mu)\, d\mu = -\varphi_0(0) \int_0^1 \mu\varphi(\mu)\, d\mu = -\sqrt{D_0}\,\varphi_0(0).$$

The angular distribution function on the boundary $\psi(0, -\mu) = \varphi(\mu)\psi_0(0)$ satisfies the equation

$$\int_0^1 \frac{y\varphi(y)}{y+\mu}\, dy = \frac{p}{2\varphi(\mu)(1-k^2\mu^2)}. \qquad (9)$$

Eqs. (8) and (9) make it possible to obtain approximate expressions for the function $\varphi(\mu)$. In the zero approximation the angular distribution is approximated by the linear rational function

$$\varphi_0(\mu) = \frac{a+b\mu}{1-k^2\mu^2},$$

and the constants a and b are found from the normalization condition and the exact value of the flux on the boundary

$$\int_0^1 \varphi_0(\mu)\, d\mu = 1; \quad \int_0^1 \mu\varphi_0(\mu)\, d\mu = \frac{\sqrt{1-p}}{k}.$$

We find that

$$\varphi_0(\mu) = b\,\frac{\frac{\sqrt{1-p}}{k} + \mu}{1-k^2\mu^2},$$

$$\frac{1}{b} = \frac{1}{2k^2}\ln\frac{1}{1-k^2} + \frac{\sqrt{1-p}}{kp}. \qquad (10)$$

** (see previous page)
At $p > 1$ k is imaginary and the solution becomes periodic; the author made a detailed derivation of the subsequent equations in 1951.

In a particular case, when p=1, Eq.(10) becomes the angular distribution formula put forward by Fermi

$$\varphi(\mu)=\frac{1+\sqrt{3}\mu}{1+\frac{\sqrt{3}}{2}}.$$

The following approximation can be obtained by iteration from Eq.(9)

$$\varphi_1(\mu)=\frac{p}{2(1-k^2\mu^2)\int_0^1\frac{\mu'\varphi_0(\mu')}{\mu+\mu'}d\mu'}.$$

The function $\varphi_1(\mu)$ has the exact value $\frac{p}{2}$ at $\mu=0$; at other values μ the accuracy in determining $\varphi(\mu)$ by the above method is about 0.1%.

The value $z_0(p)$ is found from the formula

$$e^{kz_0}=\sqrt{\frac{k^2-1+p}{2k^2(1-k^2)}}\cdot\frac{1}{\int_0^1\frac{\mu\varphi(\mu)}{1+k\mu}d\mu}. \qquad (11)$$

The function $z_0(p)$ can be represented with a good degree of accuracy by the formula

$$z_0(p)=\frac{0.71}{p}. \qquad (12)$$

The problem of reflection of particles from an infinite space is solved by the Wiener-Hopf method. Here we find the albedo and angular distribution of the particles escaping from the medium $\varphi_-(\mu)$ from the given angular distribution of

incident particles $\psi_+(\mu)$. The integral relationship linking $\psi_+(\mu)$ and $\psi_-(\mu)\,(0<\mu<1)$ takes the form

$$\psi_-(\mu)=\frac{2}{p}\varphi(\mu)(1-k\mu)\int_0^1\frac{\psi_+(\mu_0)(1-k\mu_0)\mu_0}{\mu+\mu_0}\varphi(\mu_0)\,d\mu_0. \tag{13}$$

in which the function $\varphi(\mu)$ is the angular distribution in Milne's problem and satisfies Eq. (9). Equation (13) has been derived by Ambartsumyan /6,7/ and Chandresekhar /8/ using other methods.

Using the integral relationship (13) we can calculate the albedo for different cases of incident angular distribution: (a) for a flux from a uniform plane isotropic source situated on the boundary $\left(\psi_+(\mu)=\frac{1}{\mu}\right)$,

$$A=\frac{\int_0^1\psi_-(\mu)\mu\,d\mu}{\int_0^1\psi_+(\mu)\mu\,d\mu}=\frac{1-\sqrt{1-p}}{1+\sqrt{1-p}}; \tag{14}$$

and (b) for isotropic distribution of the incident flux $(\psi_+(\mu)=2)$

$$A=1-\frac{4(1-p)}{pk}+\frac{4\sqrt{1-p}}{p}\int_0^1\varphi(\mu)\mu^2\,d\mu; \tag{15}$$

and (c) for angular distribution of the flux impinging normally to the surface $[\psi_+(\mu)=\delta(\mu-1)]$,

$$A=1-\frac{2}{p}(1-k)\sqrt{1-p}\,\varphi(1). \tag{16}$$

Solutions of Kinetic Equation for Two ~~Bounding~~ Oscillating Semi-infinite Media

Let us use Eq. (13) to derive a system of integral equations for the angular distribution on the plane interface between two media. Let the particle flux be directed from medium 1 to medium 2. The angular distribution of these particles will be designated as $\psi_+(\mu)$, and the angular distribution of particles leaving medium 2 and impinging on medium 1 will be designated as $\psi_-(\mu)$. Since on account of the conditions of the problem the particle density in medium 2 tends to zero at $z \to \infty$, the connection between the functions $\psi_+(\mu)$ and $\psi_-(\mu)$ is determined by Eq. (13)

$$\psi_-(\mu) = \frac{2}{p_2}\varphi_2(\mu)(1 - k_2\mu)\int_0^\infty \frac{\psi_+(\mu_0)(1 - k_2\mu_0)\mu_0}{\mu + \mu_0}\varphi_2(\mu_0)\,d\mu_0, \qquad (13')$$

in which the subscript 2 means that the corresponding values and functions refer to medium 2. The other equation is similar in form, but it must be taken into account that the particle density increases exponentially in medium 1 at $z \to -\infty$, and that in order to use Eq. (13) we have to separate from $\psi_+(\mu)$ the part of the angular distribution $C\varphi_1(\mu)$ responsible for the exponential increase in the particle density at $z \to -\infty$. Thus,

$$\psi_+(\mu) - C\varphi_1'(\mu) = \frac{2}{p}\varphi_1(\mu)(1-k_1\mu)\times$$

$$\times\int_0^1 \frac{\psi_-(\mu_0)(1-k_1\mu_0)\mu_0}{\mu+\mu_0}\varphi_1(\mu_0)\,d\mu_0, \qquad (17)$$

in which C is the normalizing constant determining the flux coming from medium 1.

The solution of the system of two equations (13′) and (17) takes the form

$$\psi_+(\mu) = Bp_2\frac{\varphi_1(\mu)}{(1-k_2\mu)\varphi_2(\mu)},$$

$$\psi_-(\mu) = Bp_1\frac{(1-k_2\mu)\varphi_2(\mu)}{(1-k_1^2\mu^2)\varphi_1(\mu)}, \qquad (18)$$

which can be seen by direct substitution. The normalizing constant B is determined, for example, from the condition that the density on the media boundary is equal to unity

$$\int_0^1 [\psi_+(\mu) + \psi_-(\mu)]\,d\mu = 1.$$

In the given case $B = \frac{1}{2}$, while the flux on the boundary is expressed by

$$J = \int_0^1 \mu[\psi_+(\mu) - \psi_-(\mu)]\,d\mu = \frac{\sqrt{(1-p_1)(1-p_2)}}{k_1}. \qquad (19)$$

The angular distribution on the boundary and the flux are calculated by different methods in /9,10/.

We should point out that in a number of cases equations of

type (13′) and (17) can be solved if the solution is sought in the form

$$\psi_+(\mu) = R_1(\mu) + R_2(\mu)\frac{\varphi_1(\mu)}{\varphi_2(\mu)},$$
$$\psi_-(\mu) = R_3(\mu) + R_4(\mu)\frac{\varphi_2(\mu)}{\varphi_1(\mu)}, \tag{20}$$

in which $\underline{R}(\mu)$ are linear rational functions.

The coefficients in the linear rational functions are found from algebraic relationships derived after substitution of expressions (20) into the initial equations.

The particle density function $\psi_0(z)$ in the two media problem, as in Milno's problem, can be represented as

$$\psi_0(z) = C_{1-}e^{-k_1 z} + C_{1+}e^{k_1 z} + f_1(z) = n_1(z) + f_1(z) \quad (z<0);$$
$$\psi_0(z) = C_{2-}e^{-k_2 z} + f_2(z) = n_2(z) + f_2(z) \quad (z>0),$$

in which $n(z)$ is the asymptotic density and $\underline{f}(z)$ only differs from zero in direct proximity to the boundary. The asymptotic coefficients $\underline{C}_{1-}$, $\underline{C}_{2-}$ and $\underline{C}_{1+}$ are found as deductions with respect to the poles $\underline{s} = \pm \underline{k}_i$ of the Laplacian mode $\Phi_{1,0}(s)$ of the function $\psi_0(z)$, which in turn is expressed by the angular distribution on the boundary $\psi(0, \mu)$ using the following formulae

$$\Phi_{1,0}(s) = \frac{\int_0^1 \frac{\mu\psi_-(\mu)\,d\mu}{1+\mu s} - \int_0^1 \frac{\mu\psi_+(\mu)\,d\mu}{1-\mu s}}{1 - \frac{p_1}{s}\operatorname{Arth} s},$$
$$\Phi_{2,0}(s) = \frac{\int_0^1 \frac{\mu\psi_+(\mu)\,d\mu}{1+\mu s} - \int_0^1 \frac{\mu\psi_-(\mu)\,d\mu}{1-\mu s}}{1 - \frac{p_2}{s}\operatorname{Arth} s}, \tag{21}$$

The results of integrating can easily be shown by means of the function

$$\tau(k,s)=\frac{s^2-1}{s^2-k^2}\left(1-\frac{p(k)}{s}\operatorname{Arth} s\right) \quad \text{and}$$

$$\tau_-(k,\ s)=\frac{1}{(1+s)\int_0^1\frac{\mu\varphi(k,\ \mu)}{1+\mu s}\,d\mu}, \tag{22}$$

and

$$\tau_-(k,\ s)\tau_-(k,-s)=\frac{1}{\tau(k,\ s)},$$

$$\int_0^1\frac{\mu\varphi(k,\ \mu)\,d\mu}{1+\mu s}\int_0^1\frac{\mu\varphi(k,\ \mu)\,d\mu}{1-\mu s}=\frac{k^2-s^2}{1-\frac{p}{s}\operatorname{Arth} s}.$$

for

The coefficients are equal to

$$C_{1-}=\frac{(k_1+k_2)\tau_-(k_1,\ k_1)}{2k_1\tau_-(k_2,\ k_1)},$$

$$C_{1+}=\frac{(k_1-k_2)\tau_-(k_2,\ k_1)\tau(k_2,\ k_1)}{2k_1\tau_-(k_1,\ k_1)\tau(k_1,\ k_1)},$$

$$C_{2-}=\frac{\tau(k_1,\ k_2)\tau_-(k_1,\ k_2)}{\tau(k_2,\ k_2)\tau_-(k_2,\ k_2)}, \tag{23}$$

$$C_{2+}=0.$$

The solution of the problem corresponding to the flux directed from medium 2 to medium 1 is obviously derived by replacing the subscript 2 by 1, and vice-versa, in the first problem. The general solution represents a linear combination of the two solutions, and in the general solution the asymptotic density satisfies the following boundary conditions*: 1. Equality of the logarithmic derivatives of the asymptotic density at the extrapolated points

$$\frac{n_1'(z_1)}{n_1(z_1)}=\frac{n_2'(z_2)}{n_2(z_2)}. \tag{24}$$

* This problem was solved by D. Zaretskiy as well (1952) after he had read the author's work (1951).

2. Discontinuity of the asymptotic density at the extrapolated points

$$\frac{n_1(z_1)}{\tau(\rho_1)} = \frac{n_2(z_2)}{\tau(\rho_2)} . \tag{25}$$

The functions

$$\tau(\rho) = \sqrt{\frac{2k^2(1-k^2)}{3\rho(k^2-1+\rho)}} \tag{26}$$

are given in Table 1.

The position of the extrapolated points $\underset{\wedge}{z}_1$ and $\underset{\wedge}{z}_2$ in Eqs. (24) and (25) are found by the following formulae

$$\begin{aligned} z_1 &= f(k_1, k_1) - f(k_2, k_1), \\ z_2 &= f(k_1, k_2) - f(k_2, k_2), \end{aligned} \tag{27}$$

in which z is the distance from the extrapolated point to the interface; [have,] if z is positive, the extrapolated point is in medium 2 (its parameters have a subscript 2), whereas if it is negative then it is found in medium 1. The extrapolated length z_0 (Eq. 11) is also linked to the function f, to wit: $z_0 = f(k, k)$. The function $f(k,s)$ from Eq. 5 is determined from one of the equivalent formulae

$$\operatorname{ch} sf(k, s) = \sqrt{\frac{p(s)(k^2 - s^2)}{p(s) - p(k)}} \int_0^1 \frac{\mu\varphi(k, \mu)}{1 - s^2\mu^2}\, d\mu;$$

$$\operatorname{sh} sf(k, s) = s\sqrt{\frac{p(s)(k^2 - s^2)}{p(s) - p(k)}} \int_0^1 \frac{\mu^2\varphi(k, \mu)}{1 - s^2\mu^2}\, d\mu. \qquad (28)$$

An approximation of Eq. 10 can be used as $\varphi(k, \mu)$; $p(k)$ is the reciprocal function of function $k(p)$ determined from the transcendental Eq. (3).

Table 2 shows the values of z. To determine their sign we can conveniently use the rule that both extrapolated points are always in the more active medium (in the medium for which p is greater), and the extrapolated point of the less active matter is further from the boundary. All the preceding derivations are applicable to a breeding medium ($p > 1$) and the corresponding formulae hold, since k is purely imaginary in this case.

To complete the picture let us write down the relationships linking the true particle density and the particle density on the boundary between two media with the asymptotic density and its derivative:

$$\varphi_0(0) = \sqrt{\frac{k_2^2 - k_1^2}{3(p_1 - p_2)}}\, \frac{n_1(z_1)}{\gamma(p_1)} = \sqrt{\frac{k_2^2 - k_1^2}{3(p_1 - p_2)}}\, \frac{n_2(z_2)}{\gamma(p_2)};$$

$$J(0) = -\frac{\sqrt{(1 - p_1)(1 - p_2)}}{k_1 k_2} \sqrt{\frac{k_2^2 - k_1^2}{3(p_1 - p_2)}}\, \frac{n_1'(z_1)}{\gamma(p_1)} = \qquad (29)$$

$$= -p_{1,2}\frac{n_1'(z_1)}{\gamma(p_2)}.$$

Solution of Kinetic Equation With External Particle Source

Equations of type (12') and (17) for determining the angular distribution on the boundary between two semi-infinite media can be

Table 2

Distance (z) between extrapolated point and interface of media *

At p for the adjoining medium

p	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
0,1	−1,4616	0,00	3,4624	4,7484	5,3551	5,7040	5,9295	6,0871	6,2040	6,2938	6,3650
0,2	−1,0756	−0,7656	0,00	0,8066	1,2867	1,5794	1,7733	1,9127	2,0139	2,0937	2,1581
0,3	−0,8365	−0,6753	−0,3811	0,00	0,3083	0,5229	0,6748	0,7868	0,8727	0,9402	0,9957
0,4	−0,6739	−0,5649	−0,4031	−0,1927	0,00	0,1510	0,2648	0,3527	0,4224	0,4784	0,5247
0,5	−0,5582	−0,4799	−0,3727	−0,2384	−0,1084	0,00	0,0964	0,1561	0,2114	0,2574	0,2962
0,6	−0,4734	−0,4146	−0,3363	−0,2419	−0,1483	−0,0672	0,00	0,0751	0,1108	0,1391	0,1716
0,7	−0,4093	−0,3622	−0,3027	−0,2322	−0,1614	−0,0983	−0,0452	0,00	0,0379	0,0702	0,0978
0,8	−0,3599	−0,3215	−0,2744	−0,2190	−0,1634	−0,1128	−0,0691	−0,0320	0,00	0,0275	0,0512
0,9	−0,3208	−0,2893	−0,2503	−0,2035	−0,1603	−0,1189	−0,0825	−0,0511	−0,0238	0,00	0,0205
1,0	−0,2887	−0,2615	−0,2292	−0,1923	−0,1547	−0,1200	−0,0892	−0,0623	−0,0388	−0,0181	0,00
1,1	−0,2623	−0,2390	−0,2112	−0,1801	−0,1484	−0,1187	−0,0925	−0,0691	−0,0487	−0,0306	−0,0144
1,2	−0,2403	−0,2198	−0,1950	−0,1693	−0,1421	−0,1166	−0,0936	−0,0732	−0,0552	−0,0391	−0,0246
1,3	−0,2216	−0,2036	−0,1826	−0,1596	−0,1358	−0,1136	−0,0934	−0,0753	−0,0593	−0,0449	−0,0319
1,4	−0,2056	−0,1895	−0,1721	−0,1507	−0,1299	−0,1103	−0,0924	−0,0762	−0,0618	−0,0489	−0,0371
1,5	−0,1914	−0,1769	−0,1610	−0,1425	−0,1240	−0,1065	−0,0905	−0,0760	−0,0631	−0,0513	−0,0406
1,6	−0,1794	−0,1666	−0,1517	−0,1356	−0,1193	−0,1033	−0,0889	−0,0759	−0,0640	−0,0534	−0,0436
1,7	−0,1686	−0,1566	−0,1433	−0,1291	−0,1141	−0,1000	−0,0869	−0,0750	−0,0642	−0,0545	−0,0455
1,8	−0,1592	−0,1457	−0,1360	−0,1230	−0,1095	−0,0967	−0,0848	−0,0740	−0,0641	−0,0552	−0,0467
1,9	−0,1505	−0,1403	−0,1298	−0,1175	−0,1052	−0,0935	−0,0826	−0,0727	−0,0636	−0,0553	−0,0476
2,0	−0,1429	−0,1340	−0,1240	−0,1127	−0,1014	−0,0905	−0,0806	−0,0715	−0,0630	−0,0553	−0,0482

* When using the table it should be remembered that the extrapolated point is always in the medium with a large p.

cont. Table 2.

p	At p for the adjoining medium										
	1,0	1,1	1,2	1,3	1,4	1,5	1,6	1,7	1,8	1,9	2,0
0,1	6,3880	6,4247	6,4736	6,5150	6,5505	6,5808	6,6082	6,6321	6,6534	6,6725	6,6897
0,2	2,1581	2,2107	2,2547	2,2921	2,3243	2,3523	2,3769	2,3987	2,4181	2,4356	2,4512
0,3	0,9967	1,0413	1,0797	1,1126	1,1410	1,1658	1,1878	1,2073	1,2248	1,2405	1,2548
0,4	0,5247	0,5634	0,5964	0,6248	0,6496	0,6714	0,6907	0,7080	0,7235	0,7376	0,7503
0,5	0,2962	0,3290	0,3572	0,3817	0,4032	0,4223	0,4393	0,4546	0,4684	0,4809	0,4924
0,6	0,1716	0,1994	0,2237	0,2450	0,2638	0,2805	0,2955	0,3091	0,3214	0,3326	0,3429
0,7	0,0978	0,1219	0,1429	0,1615	0,1779	0,1927	0,2060	0,2182	0,2292	0,2393	0,2485
0,8	0,0512	0,0721	0,0905	0,1068	0,1214	0,1346	0,1465	0,1575	0,1674	0,1764	0,1848
0,9	0,0205	0,0387	0,0549	0,0694	0,0825	0,0942	0,1050	0,1148	0,1238	0,1319	0,1395
1,0	0,00	0,0160	0,0305	0,0435	0,0552	0,0659	0,0756	0,0846	0,0927	0,1003	0,1073
1,1	−0,0144	0,00	0,0130	0,0247	0,0353	0,0449	0,0538	0,0619	0,0694	0,0763	0,0825
1,2	−0,0246	−0,0116	0,00	0,0106	0,0203	0,0290	0,0371	0,0446	0,0514	0,0578	0,0538
1,3	−0,0319	−0,0203	−0,0096	0,00	0,0088	0,0168	0,0243	0,0312	0,0375	0,0442	0,0490
1,4	−0,0371	−0,0265	−0,0168	−0,0031	0,00	0,0074	0,0143	0,0206	0,0265	0,0320	0,0372
1,5	−0,0406	−0,0309	−0,0221	−0,0140	−0,0065	0,00	0,0067	0,0125	0,0180	0,0231	0,0279
1,6	−0,0436	−0,0347	−0,0266	−0,0191	−0,0122	−0,0059	0,00	0,0055	0,0106	0,0155	0,0199
1,7	−0,0455	−0,0373	−0,0298	−0,0229	−0,0165	−0,0105	−0,0051	0,00	0,0047	0,0093	0,0135
1,8	−0,0467	−0,0393	−0,0324	−0,0260	−0,0200	−0,0144	−0,0094	−0,0047	0,00	0,0042	0,0082
1,9	−0,0476	−0,0406	−0,0342	−0,0282	−0,0226	−0,0174	−0,0126	−0,0092	−0,0040	0,00	0,0037
2,0	−0,0482	−0,0417	−0,0356	−0,0301	−0,0249	−0,0200	−0,0155	−0,0114	−0,0073	−0,0036	0,00

used to solve the problem of the particle distribution around an isotropic source located on the boundary of these media. In this case the equations take the form

$$\varphi_-(\mu) = \frac{2}{p_1}\varphi_1(\mu)(1-k_1\mu)\int_0^1 \frac{\mu_0\varphi_+(\mu_0)(1-k_1\mu_0)\varphi_1(\mu_0)}{\mu+\mu_0}d\mu_0 + \frac{1}{2p_1};$$

$$\varphi_+(\mu) = \frac{2}{p_2}\varphi_2(\mu)(1-k_2\mu)\int_0^1 \frac{\mu_0\varphi_-(\mu_0)(1-k_2\mu_0)\varphi_2(\mu_0)}{\mu+\mu_0}d\mu_0 + \frac{1}{2p_2}. \qquad (30)$$

The source on the boundary is normalized in such a way that the total flux from a unit of surface is equal to unity. Equations (30) are solved by means of Eq. 20, and the functions $\varphi_+(\mu)$ and $\varphi_-(\mu)$ take the following form

$$\varphi_+(\mu) = \frac{p_1(1-k_2\mu)\varphi_2(\mu)}{2(p_2-p_1)\mu(1-k_2\mu)\varphi_1(\mu)} + \frac{p_1}{2(p_1-p_2)\mu};$$

$$\varphi_-(\mu) = \frac{p_2(1-k_1\mu)\varphi_1(\mu)}{2(p_1-p_2)\mu(1-k_2\mu)\varphi_2(\mu)} + \frac{p_2}{2(p_2-p_1)\mu}. \qquad (31)$$

Just as before, the particle density is represented as the sum of the exponential asymptotic density and the addition

$$\varphi_0(z) = C_{1+}e^{k_1 z} + f_1(z) \qquad (z<0);$$
$$\varphi_0(z) = C_{2-}e^{-k_2 z} + f_2(z) \qquad (z>0).$$

On account of the condition of the problem, that there is no flux from inifinity, the particle density at $z \to \infty$ and $z \to -\infty$ tends exponentially to zero, i.e. $C_{1-} = C_{2+} = 0$. Just as before, the asymptotic coefficients are found from Eqs. 21 and are equal to

$$C_{1+} = \frac{(k_1 - k_2)\tau(k_2, k_1)\tau_-(k_2, k_1)}{(p_2 - p_1)\tau(k_1, k_1)\tau_-(k_1, k_1)} =$$
$$= \sqrt{\frac{3(k_1 - k_2)}{(k_1 + k_2)(p_2 - p_1)}}\gamma_1 e^{-k_1 z_1};$$
$$C_{2-} = \frac{(k_2 - k_2)\tau(k_1, k_2)\tau_-(k_1, k_2)}{(p_2 - p_1)\tau(k_2, k_2)\tau_-(k_2, k_2)} =$$
$$= \sqrt{\frac{3(k_1 - k_2)}{(k_1 + k_2)(p_2 - p_1)}}\gamma_2 e^{-k_2 z_2}, \qquad (32)$$

in which z_1 and z_2 are the distances from the extrapolated points to the interfaces determined by 27.

In the special case of a homogenous medium

$$C_{1-} = C_{2+} = \frac{3}{2k}\gamma^2$$
$$\psi_0(z) = \frac{k(1-k^2)}{p(k^2 - 1 + p)} e^{-k|z|} + f(z). \qquad (33)$$

and

This partial solution is contained in Marshak's review /11/.

The function $f(z)$ is positive everywhere, as a logarithmic anomaly at $z = 0$ and decreases rapidly as z increases. The integral of function $f(z)$ can be found by using the law of conservation[2] of the total number of particles $(1-p)\int_{-\infty}^{+\infty}\psi(z)dz = 1$, from which

$$\int_{-\infty}^{+\infty} f(z)\,dz = \frac{1 - 3D_0\gamma^2}{1-p}. \qquad (34)$$

Solution of Kinetic Equation for a Sphere

When solving spherical problems in which the particle path length is constant, it is very useful indeed to use the theorem

of the reduction of a spherical problem to a plane one*. According to this theorem, which can easily be derived if we write down the kinetic equations in the form of the Peierls equation, a spherical problem with a set distribution of the function $\underline{p}(\underline{r})$ is equivalent to a plane problem in which $\underline{p}(\underline{x}) = \underline{p}(-\underline{x}) = \underline{p}(\underline{r})$. To obtain the particle density in a spherical case, we select an odd solution of the plane problem $\psi_0(x)$, and the particle density in the spherical problem $\psi_0(r)$ is expressed in terms of the particle density in the plane problem by the formula

$$\psi_0(r) = -\frac{\psi_0(x)|_{x=r}}{r}. \tag{35}$$

Here, of course, the angular distributions do not change into one another.

This theorem implies a number of practical corollaries:

1. In an infinite spherical problem the asymptotic solution takes the form

$$n(r) = C_1\frac{e^{\nu r}}{r} + C_2\frac{e^{-\nu r}}{r}, \tag{36}$$

the asymptotic density in the spherical problem, just as in the plane problem, is the solution of the diffusion equation (4)

with a corrected asymptotic diffusion coefficient.

2. When deriving boundary conditions in an equi-path spherical problem, $\underline{nr}$ has to be substituted into the boundary conditions of the plane problem instead of function $\underline{n}$. The spherical problem with a constant total path length, in which

$$\left.\begin{aligned} p(r) &= p_1, \quad r < R \\ p(r) &= p_2, \quad r > R \end{aligned}\right\} \tag{37}$$

reduces to a plane three-layer problem

$$\begin{aligned} p(x) &= p_2, \quad x < -R \ \text{и} \ x > R; \\ p(x) &= p_1, \quad -R < x < R. \end{aligned} \tag{38}$$

Here the boundary conditions imposed on the asymptotic density on the boundaries $\pm \underline{R}$ are the same as in the problem for two semi-infinite media (Eqs. 24 and 25)). When satisfying the conditions of applicability of this approximation the true particle density on the boundary is related to the asymptotic density by the following equation, easily obtained from (29),

$$\varphi_0(R) \doteq \sqrt{\frac{k_2^2 - k_1^2}{3(p_1 - p_2)}}\, \frac{n(R + z_1)(R + z_1)}{R}. \tag{39}$$

Derivation of the flux on the boundary is more complex, since the theorem of reduction of the spherical problem to a plane one

when the path length is constant is not applicable to the flux. The particle density function may be represented as the sum of the asymptotic density $\underline{n}(\underline{z})$ and the additive $\underline{f}(\underline{z})$. In the plane problem the particle flux on the boundary ($\underline{z} = 0$) is expressed in terms of the integral of function $\underline{f}(\underline{z})$ in the following way

$$j(0) = -D_{0,1}\frac{dn_1}{dz}\Big|_{z=0} - (1-p_1)\int_{-\infty}^{0} f_1(z)\,dz. \qquad (40)$$

On the other hand, Eq. (29) gives us

$$j(0) = -P_{1,2}\frac{n_1'(z_1)}{\tau(p_1)}.$$

The relationship for the flux, similar to Eq. (40), takes the form

$$j(R) = -D_{0,1}\frac{dn_1}{dr}\Big|_{r=R} - \frac{(1-p_1)}{R^2}\int_{0}^{R} f_1(r)\,r^2 dr. \qquad (40')$$

If the conditions of applicability of the approximation are satisfied, by virtue of the theorem of reduction we obtain the relationship $f_1(r)r = f_1(z)$. If the total path length of the particles is constant, by expressing $\underline{f}_1(\underline{r})$ in terms of $\underline{f}_1(\underline{z})$ in Eq. 40' and substituting the integrals of $\underline{f}$ [$\underline{f}_1(\underline{z})\underline{dz}$ is found from Eqs. 40 and 29, while $\int f_1(z)\,z\,dz$ is calculated by differentiation at point $\underline{s} = 0$

of the Laplacian modes of the particle density in the plane problem (21)/, we find an expression for the flux in the spherical problem

$$j(R) = -\frac{P_{1,2}}{R_T(p_1)}\left[\frac{d(nr)}{dr}\Big|_{r=R+z_1} - \frac{nr}{R}\Big|_{r=R+z_1} + \right.$$
$$\left. + \frac{1}{R}\frac{d(nr)}{dr}\Big|_{r=R+z_1}\left(f(k_i, 0) - f(k_e, 0)\right)\right]. \qquad (41)$$

The functions $f(k,s)$ are determined by Eqs. 28; in (41) these functions are taken at $s = 0$. The subscripts i and e mean that the corresponding parameter refers to the internal or external medium. Eq. (41) obviously still holds when subscript 1 is replaced by subscript 2 by virtue of the boundary conditions.

Use of Exact Solutions to Calculate Diffusion Problems

(Improved Diffusion Method)

The approximate method of calculating multi-layer plane and spherical problems for media with a constant total particle path length can be reduced to the following: an asymptotic particle density function, satisfying the diffusion equation with an asymptotic diffusion coefficient, is introduced in each layer, and boundary conditions obtained from/exact solution of the two media problem are imposed upon it at the boundaries at the extrapolated points. The boundary conditions in the plane problem/ Eqs. (24) and (25)/ can

be reduced to the equality of logarithmic derivatives and discontinuity in asymptotic density at the extrapolated points near the boundary. As is clear from a survey of American research on the kinetic theory of neutrons /12/, a similar method called the boundary point method has been worked out and is being used in American calculations.

Questions of the accuracy, limits of applicability and the possibility of using the method to solve spherical problems in media with a varying total particle path length, in cylindrical problems and problems with distributed sources of particles can best be illustrated by solving specific problems.

Critical Dimensions of a Plate and Sphere

If the origin of the coordinates is selected in the center of a plate, the asymptotic particle density in the plate is expressed by the function

$$n(z) = B\cos kz^{*}.$$

The asymptotic density at a point, the distance of which from the boundary is the distance of the extrapolated length, is equal to zero

* For Purposes of convenience k will mean the imaginary part of the transcendental equation root from now on at p > 1

The critical thickness of the plate is

$$\kappa\left(\frac{\Delta}{2}+z_0\right)=B\cos k\left(\frac{\Delta}{2}+z_0\right)-0. \tag{42}$$

We should point out that the same result is found if we use condition (24) for the equality of logarithmic derivatives at the extrapolated point as the boundary conditions. Since there is no reflection we can assume $\underline{p}$ = 0 beyond the plate. Hence, $\underline{k}$ = 1 and the logarithmic derivative of the asymptotic function in the ficticious medium beyond the plate is already equal to -1. According to condition (24), the logarithmic derivative of the asymptotic density in the plate is equal to -1 at the extrapolated point z, i.e.,

$$k\operatorname{tg} k\left(\frac{\Delta}{2}+z\right)=1. \tag{42'}$$

At $\underline{p}$ = 0 / at $\underline{p}$ = 0 $\varphi(\mu)=\delta(\mu-1)$ /, Eqs. (27) and (28) give us

$$z=z_0-\frac{1}{\beta}.$$

After which it is clear both critical conditions (42) and (42') are equivalent. In order to determine the degree of accuracy of the formula for critical dimensions of the plate, Table 3 compares

calculations of the critical thickness by the variation method. A formula for the critical dimensions of an active sphere can be obtained in similar fashion

$$R = \frac{\pi}{k} - z_0 = \frac{\pi}{k} - \frac{0{,}71}{p} \qquad (43)$$

/provided z_0 is represented by Eq. 12/.

Table 2

Values of Critical Thicknesses of the Plane Layer Without Reflector

A	1/p. From eq. (42)	1/p. Variation method	A	1/p From eq. (42)	1/p Variation method
0,1	0,142	0,157	0,6	0,482	0,48
0,2	0,242	0,251	0,8	0,560	0,56
0,4	0,388	0,39	1,0	0,590	0,59

Comparison with numerical calculations shows that when determining the critical radius, the inaccuracy of Eq. (43) does not exceed 1% anywhere over the range of p. The high degree of inaccuracy in the given case is due to the fact that the critical radius as calculated by (43) at $p \to \infty$ changes successfully to the limiting formula quoted in /13/.

Critical Dimensions of Active Sphere Surrounded by Spherical Layer of Inert Reflector

Let us consider an active sphere surrounded by a spherical

layer of inert reflector. Let us assume that the particle path length in the reflector is equal to the total length of the particles in the sphere.

The asymptotic density in the active sphere ($p > 1$) is equal to

$$n_1 = A\frac{\sin kr}{r}.$$

At $r = 0$ the function n_1 is finite. In the reflector ($p = 1$)

$$n_2 = B + \frac{C}{r}.$$

The following boundary conditions are imposed upon the functions n_1 and n_2:

a) the function n_2 disappears on the external extrapolated boundary $R_e + 0.71$

$$n_2(R_e + 0,71) = B + \frac{C}{R_e + 0,71} = 0;$$

b) on the boundary of the active sphere ($r = R$) at the extrapolated points the logarithmic derivatives of the function nr are equal to

$$\left.\frac{d(n_1 r)}{dr}\right|_{r=R+z_0} = \left.\frac{d(n_2 r)}{dr}\right|_{r=R+z_0}$$

Since both extrapolated points lie inside the active sphere (in the sphere $\underline{p}$ is greater than in the reflector), $\underline{z}_1$ and $\underline{z}_2$ are negative.

As a result we obtain a formula for the critical dimensions

$$R = \frac{\pi}{k} - \frac{1}{k}\operatorname{arc\,tg}\, k(R_e - R + 0{,}71 - z_2) - z_1. \qquad (44)$$

To illustrate the accuracy and limits of applicability of Eq. 44 let us compare the critical radii calculated by (44) and calculated by Peierls' equation for $\underline{p}$ = 1.724 ($\underline{k}$ = 1.856, $\underline{z}_1$ = 0.0458, $\underline{z}_2$ = -0.0865), at different values of the relative thickness of the reflecting shell $\delta = \frac{R_e - R}{R}$ (Table 4).

Table 4

Critical radius of active sphere as function of reflecting shell thickness ($\underline{p}$ = 1.724)

δ	R		
	From equation (44)	from numerical calculations	from equation (45)
∞	1,207	—	1,281
5	1,097	1,106	—
3	1,046	1,048	—
2	0,995	1,003	—
1	0,955	0,954	—
0,5	0,937	0,94	—
0	0,921	0,93	—

Multiplication Factor for Point Source in Homogeneous Sphere

The asymptotic density in a sphere can be represented as the sum of the homogeneous and inhomogeneous solutions of Eq. 30.

$$n(r)=\frac{3}{4\pi}\gamma^2\frac{e^{-kr}}{r}+\frac{C\,\mathrm{sh}\,kr}{r}.$$

On function $n(r)$ we imposed a boundary condition $n\left(R+\frac{0,71}{p}\right)=0$ from which

$$n(r)=\frac{3}{4\pi}\gamma^2\frac{\mathrm{sh}\,k\left(R+\frac{0,71}{p}-r\right)}{r\,\mathrm{sh}\,k\left(R+\frac{0,71}{p}\right)}.$$

The flux through the external surface $r = R$ is found from Eq. (41). As a result we obtained the multiplication factor Q which is equal to the ratio of the total particle flux emerging through the external surface and the total particle flux of the source

$$Q=\sqrt{\frac{3(1-p)}{p}}\,\gamma\,\frac{R+f(k,0)}{\mathrm{sh}\,k(R+x_0)}. \tag{45}$$

Equation (45) gives us the mean path length of the particles in the sphere

$$\bar{l}=\lim_{p\to 1}\frac{Q-1}{p-1}=-0,1+\frac{0,54}{R+0,71}+\frac{1}{2}(R+0,71)^2. \tag{46}$$

At $R > 1$ the error in Eq. (46) does not exceed 1% and at $R = 0$ it gives an incorrect limiting value, and at low R we can obtain $\bar{l}$ by the successive collision method.

Application of Method To Solving Spherical Problems In Media which Total Particle Path Length is Not Constant

The main difficulty in solving the spherical problem for media with different total free path lengths is determining the conditions which have to be imposed on the asymptotic density at the interface . It has not been possible to find analytically exact solutions for this case, but the derivation of boundary conditions from numerical solutions requires complex calculations. Boundary coniditions obtained by numerical calculations are only found in scientific literature for a few individual cases (absolutely absorptive spheres and cylinders in an inert medium).

In one of his research projects the author attempted to compile boundary conditions for the spherical problem with a non-constant particle path length by introducing a further discontinuity in the neutron density on the interface proportional to the flux and different in path length. But the boundary conditions when corrected in this way have a small sphere of applicability and do not solve the set task.

The following approximate system may be put forward for calculating the general case of a varying path length. When there

are no sources, the following diffusion equation is satisfied throughout

$$\operatorname{div} l \operatorname{grad} D_0 n + (p-1) ln = 0, \tag{47}$$

in which $\underline{l}$ is the total path length (or transport-length in non-isotropic scattering, in the laboratory coordinate system). The equation contains boundary condition for n on the discontinuity boundaries 1 and p:

a) continuity of the flux on the boundary

$$l_1 \operatorname{grad} D_{0,1} n_1 = l_2 \operatorname{grad} D_{0,2} n_2; \tag{48}$$

b) discontinuity in particle density:

$$D_{0,1} n = D_{0,2} n. \tag{48'}$$

Conditions (48), (48') at $\underline{l}_1 = \underline{l}_2$ differ from the exact ones for this case, conditions (24) and (25). In Eqs. (48) and (48') the boundary conditions are imposed at the true boundaries, and not at the extrapolated points, and the density discontinuity at the boundary is $\frac{D_{0,1}}{D_{0,2}}$, and not $\frac{\gamma_2}{\gamma_1}$. Since $\underline{z}$ is usually considerably smaller than the path length, and the function $D_{0\gamma}$ is almost constant over a wide

range of variation in p, this approximation proves satisfactory in a number of cases. Since z is maximum on the external boundary, the exact boundary conditions are best kept there . Instead of conditions (48'), the following conditions are often imposed at the boundary

$$\frac{n_1}{\gamma_1} = \frac{n_2}{\gamma_2}.$$

These conditions are not a corollary of the differential equation (47), but are more exact in a number of cases. It should be pointed out that in a case in which the path length in the outside layer of the spherical system tends to infinity, conditions (48) and (48) do not give a correct limiting process.

Problem with Distributed Source in Sphere

To solve the problem using the thoery of pertubations*, let us replace the true distribution of sources $q(r)$ by an equivalent $(r) = \xi\phi_0(r)$ the value ξ being found from condition

$$\xi \int \phi_0^2 dV = \int q\phi_0 dV. \qquad (49)$$

After this the problem to find the critical value of the parameter $p' = p + \xi$ at a known value of the sphere radius R /from Eq. (43)/. $\psi_0(r)$ is the eigenfunction in the critical state. From

* In the general case of varable Peierls parameters and distribution of the

(see next page)

isotropic sources q, the averaging conditions for the theory of perturbation take the form

$$\int(\beta-a)n^2dV+\int qndV=[(\overline{\beta}-\overline{a})+\overline{a\xi}]\int n^2dV;$$
$$\int af^2dV=\overline{a}\int f^2dV$$

($\underline{f}$ is the density of the particle flux). The condition for averaging the parameter is written down in the diffusion approximation. The theory of perturbation for a kinetic equation was developed by Fuchs and, independently, in the USSR by N.A. Dmitriyev (1948).

the condition of particle balance we find the outward flux and the multiplication factor which is equal to

$$Q = \frac{4\pi R^2 j}{\int q dV} = 1 - \frac{(1-p)\int \psi_0 dV \int q\psi_0 dV}{(p'(R)-p)\int q dV \int \psi_0^2 dV} \qquad (50)$$

Eq. (50) is exact near the critical state, but its accuracy is satisfactory in other cases as well, even at p = 0 (absolutely absorptive medium).

For a source distributed throughout the eigenfunction, Eq. (50) is exact for any values of R, and the multiplication factor is

$$Q = 1 + \frac{p-1}{p'(R)-p}.$$

Below we give formulae determining the eigenfunctions and its integrals for the case of an active sphere. The integral $\int \psi_0^2 dV$ in the critical state is expressed, according to the perturbation theory , in terms of the derivative of the critical radius with respect to the parameter p

$$\int \psi_0^2 dV = -p\psi_0^2(R) 4\pi R^2 \frac{\partial R}{\partial p}.$$

The integral is found from the particle number balance; thus,

$$\int \psi_0 dV = \frac{4\pi R^2 j(R)}{p-1}.$$

For an active sphere in the critical state, Eqs. (39) and (41) give us the following relationships

$$\frac{f(R)}{\psi_0(R)} = \sqrt{D_0}\,\frac{R + f(k, 0)}{R},$$

$$\frac{\psi(0)}{\psi(R)} = \sqrt{3\rho}\,R_T.$$

For a source with a constant density, we obtain from Eq. (50)

$$Q = 1 + \frac{3D_0(\rho')(\rho - 1)}{\rho'(\rho' - 1)^2\,(\rho'(R) - \rho)\left(-\frac{dR}{d\rho}\right)_{\rho = \rho'(R)}} \times \times \frac{[R + f(k', 0)]^2}{R^2}.$$

For a central point source, Eq. (50) gives us a formula coinciding near the critical state with (45).

This method of calculation was then applied to the solution of a number of specific problems.

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Approximate Method of Calculating Critical Masses of Spherical Reactors with Infinite Reflectors

by G. I. Marchuk and V. P. Kochergin

Let us consider a critical reactor in which the active zone of the radius R, which is a spherically symmetrical neutron source, is surrounded by an infinite reflector; there are no neutron sources in the reflector. If the origin of the coordinates is selected in the center of the active zone, the solution of the single-group diffusion equation for the neutron flux in the reflector takes the form /1/

$$\Phi(r) = C\frac{e^{-\lambda r}}{r}. \tag{1}$$

The effect of the reflector on reducing the critical dimensions of the reactor is described by the effective addition

$$\delta R = R_e - R, \tag{2}$$

in which R_e is the extrapolated radius of the reactor without a reflector.

The effective addition can also be defined as the length of the linear extrapolation of the neutron flux in the reflector on the

boundary with the reactor's active zone

$$\delta R = -\frac{\Phi}{d\Phi/dr}\Big|_{r=R} = \frac{1}{\lambda + \frac{1}{R}} \quad (3)$$

from the equality of Eqs. (2) and (3) we can determine

$$\lambda = \frac{1}{R_0 - R} - \frac{1}{R}. \quad (4)$$

If λ and $\underline{R}_{/e}$ are known, then by solving Eq. (4) with respect to $\underline{R}$ we obtain

$$R = \frac{\lambda R_0 - 2 + \sqrt{4 + (\lambda R_0)^2}}{2\lambda} \quad (5)$$

To make the radius positive, a plus sign is put before the root.

A convenient quantity describing the energy spectrum of the reflectorless reactor is the so-called cadmium fission chamber ratio

$$Cd\,R = \int_{u=15.5}^{\infty} \nu_f \Sigma_f \varphi(u)\,du \Big/ \int_{-\infty}^{u=15.5} \nu_f \Sigma_f \varphi(u)\,du. \quad (6)$$

The value λ of the function of the cadmium ratio is determined by means of the critical parameters $\underline{R}_{/e}$ and $\underline{R}$, derived by solving the multigroup diffusion equations at the given densities of the nuclear fuel and moderator, as well as at the given uranium enrichment with respect to isotope U^{235} (for example, 100%).

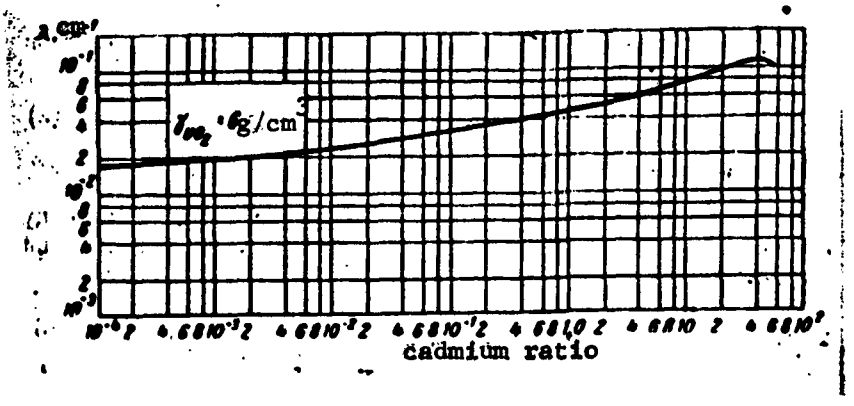

Fig. 1. Function $\lambda = \frac{1}{R_0 - R} - \frac{1}{R}$ for spherical reactors.
$UO_2 - H_2O$.

The critical size of a reactor with an infinite reflector with other enrichments and the same fuel and moderator densities is calculated from Eq. (5) using the graph for the function λ. But first a multigroup calculation of the reflector less reactors has to be made to determine R/e and the cadmium ratio.

The graphs showing the function λ for systems with water, graphite and beryllium reflectors are given in Figs. 1, 2 and 3, respectively. Comparison of the critical dimensions of the active zone for these systems, determined by means of the multigroup calculation in the papers by Barschuk and others (see pp. 39, 51 and 57 in the

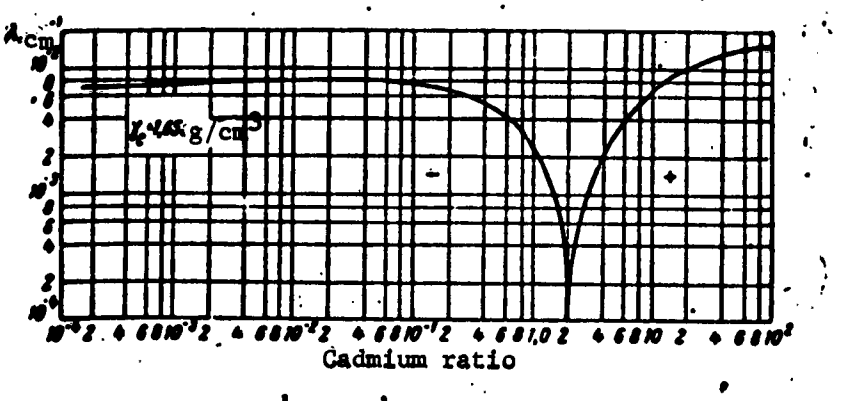

Fig. 2. Function $\lambda = \frac{1}{R_0 - R} - \frac{1}{R}$ for spherical reactors U-C.

Table 1

Critical masses of uranium-plutonium-water systems

a°	$UO_2 - H_2O$						$UO_2 - PuO_2 - H_2O$					
	$p = 35\%$			$p = 5\%$			$\beta = 0$			$\beta = 10$		
	R_e	R	R^*	R_e	R	R^*	R_e	R	R^*	R_e	R	R^*
2000	87,0	78,0	78,2	—	—	—	34,6	28,0	27,7	36,9	30,0	29,6
1200	34,6	28,0	28,0	44,4	36,8	36,8	25,4	19,0	18,7	27,8	21,0	20,8
800	27,3	21,0	20,7	34,5	27,3	27,0	—	—	—	—	—	—
600	—	—	—	—	—	—	20,8	14,5	14,2	23,6	16,5	16,3
400	22,4	15,7	15,7	30,3	22,2	22,2	—	—	—	—	—	—
200	—	—	—	—	—	—	18,4	11,8	11,6	23,9	15,5	15,5
120	20,6	13,2	13,4	39,3	27,3	28,2	18,0	11,2	11,0	26,5	17,0	17,3
60	—	—	—	68,2	52,9	52,8	—	—	—	—	—	—
30	24,3	15,0	15,1	—	—	—	18,7	11,1	11,0	52,2	35,0	36,8
10	32,5	20,0	20,4	—	—	—	19,8	11,5	11,4	—	—	—
3	41,4	25,0	25,0	—	—	—	20,8	11,6	12,0	154,0	120,0	116,0
1	—	—	—	—	—	—	21,6	11,6	11,8	99,4	66,0	67,1

Note: For the system UO_2-H_2O $a = \rho_H / \rho_U$, and for the system $UO_2-PuO_2-H_2O$ $a = \rho_H / \rho_{Pu}$, $\beta = \rho_U / \rho_{Pu}$;

P is the percentage of enrichment with isotope U^{235}

R_e and R are the extrapolated and critical radii derived by solving multigroup diffusion equations;

R^* is the critical radius calculated from Eq. (5).

Table 2

Critical masses of uranium-graphite and uranium-beryllium systems

a^o	U—C						U—Be					
	$\rho = 35\%$			$\rho = 5\%$			$\rho = 35\%$			$\rho = 10\%$		
	R_o	R	R^*	R_o	R	R^*	R_o	R	R^*	R_o	R	R^*
60000	147	106	106	176	129	132	123	100	102	140	119	119
30000	110	73,3	72,5	132	92,3	89,4	65,8	48	48,2	71,4	52,5	52,8
20000	—	—	—	122	81,7	79,1	—	—	—	—	—	—
10000	86,0	50,1	50,4	124	74,0	76,3	44,3	28,0	27,9	49,3	31,5	31,6
8000	—	—	—	129	76,0	78,0	—	—	—	—	—	—
5000	80,0	43,7	43,7	—	—	—	39,6	23,0	23,0	37,1	26,5	26,9
2000	82,0	30,9	41,6	—	—	—	37,5	19,8	19,8	49,3	25,0	25,6
1000	—	—	—	—	—	—	38,2	18,8	18,9	65,3	29,5	29,9
600	95,0	40,6	42,7	—	—	—	40,0	18,5	18,8	—	—	—
300	109,0	42,1	44,3	—	—	—	43,8	18,8	19,0	—	—	—
200	107,0	42,1	42,7	—	—	—	—	—	—	—	—	—
100	84,0	36,1	35,1	—	—	—	44,8	18,5	18,5	—	—	—
60	—	—	—	—	—	—	39,2	16,8	16,6	—	—	—
30	—	—	—	—	—	—	31,6	14,2	13,9	—	—	—

Note: For the system U—C $a = \rho_C / \rho_U$, and for the system U—Be $a = \rho_{Be}/\rho_U$. The other symbols are the same as in Table 1.

present collection) and by Eq. (5) for a wide range of variation in the ratios $a = \rho_2/\rho_{u^{235}}$, are given in Tables 1 and 2. The asterisk denotes the radius found by (5).

Discrepancies in the critical mass lie within the limits of the accuracy with which the multigroup calculation was made, and do not in effect exceed 10%.

Since λ is determined by means of the multigroup calculation, it thereby takes into account the retardation effect of the neutrons in the reflector: hence λ can be both positive and negative, according to the effectiveness of the reflector.

It follows from Eq. 5 that when λ varies between $-\infty$ and $+\infty$, $\underline{R}$ varies between 0 and $\underline{R}_e$, and

$$\lim_{\lambda \to 0} R = \frac{R_0}{2} \cdot \beta \qquad (7)$$

The value λ remains positive for the system $UO_2 - H_2O$ (γ_{UO_2} = 6 g/cm^3) within the range of variation in the cadmium ratio under consideration, and the effective addition $\delta\underline{R}$ is therefore smaller than $\underline{R}_e/2$ throughout; At a certain value of the cadmium ratio;

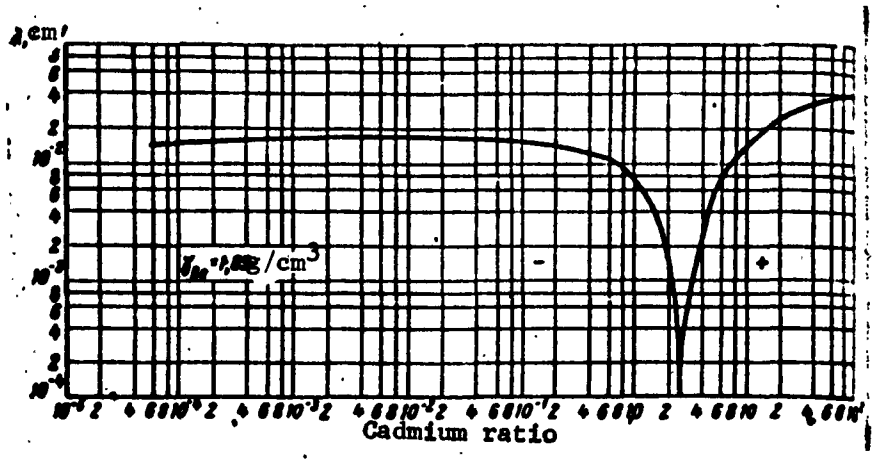

Fig. 3. Function $\lambda = \frac{1}{R_0 - R} - \frac{1}{R}$ for spherical reactors U-Be.

in the systems U - C and U - Be, λ changes its sign and becomes negative, by virtue of which the effective addition in this region becomes greater than R /2. This also follows from the Tables.

We should point out that in the calculation of the critical dimensions of the systems UO_2-PuO_2-H_2O ($\gamma_{UO_2} = \gamma_{PuO_2} = 6$ g/cm^3) from Eq. (5) it is possible to use the function λ shown in Fig. 1, and this is confirmed by Table 1, in which λ , and the uranium is natural. This fact shows that the graph of the function λ does not change when uranium is replaced by plutonium, provided the compound in which they are present exhibits the same density. It probably does not change either when uranium or plutonium are replaced by another fissile element. Thus, we are in a position to calculate the critical masses from Eq. (5) for any combinations of fissile elements in the given compound, if the graph of function λ is available.

To determine the critical masses of spherical reactors with infinite reflectors at other fuel and moderator densities, it is essential to calculate the critical parameters by means of multi-group equations for diffusion, at any enrichment of the fuel,

in order to determine the function λ.

For reflectorless reactors there exists a similarity /3/, and the critical dimensions can easily be converted when the fuel and moderator densities vary. Let us now consider a system consisting of two components.

If we fix $\alpha = p_f/p_m$, the number of fuel nuclei per 1 cm^3 is determined in the following way

$$p_f = \frac{1}{\frac{1}{p_f^*} + \alpha \frac{1}{p_m^*}}, \tag{8}$$

in which $p^* = \frac{N}{A}\gamma$ (N is the Avogadro number; γ is the density, g/cm^3; A is the atomic weight).

If γ and γ vary, p_f and p_m will vary in identical proportion, hence to change from one system to the other the macroscopic sections have to be multiplied by $\lambda = p_f^{(2)}/p_f^{(1)}$. Here, so as not to disrupt the multigroup diffusion equations, the linear dimensions have to be divided by the similarity factor δ.

Thus,

$$R_e^{(2)} = \frac{R_e^{(1)}}{\delta}. \tag{9}$$

The curves showing the critical mode as a function of the critical volume when $\gamma_{\hat{f}}$ and $\gamma_{\hat{m}}$ change in the same ratio are transposed parallelly without changing the shape, since the similarity factor δ in this case is the same for all α.

If $\gamma_{\hat{f}}$ and $\gamma_{\hat{m}}$ do not change in the same ratio, and the similarity factor therefore depends on α, the ~~load-volume~~ (showing load due to volume) curves are still more distorted in addition to ~~deformed, apart from~~ the transfer.

Footnote on page 37 of English text:

In physical meaning λ is the reciprocal of an effective diffusion length. Some authors use λ in the effective boundary conditions method /2/. In this work λ is considered from a different viewpoint.

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USE OF EVEN APPROXIMATIONS IN SPHERICAL HARMONIC METHOD

by G. Ya. Rumyantsev

When solving the kinetic Boltzmann equation by the spherical harmonic method we make use of the so-called P_N approximations, in which N is the order of approximation. In particular, the well-known diffusion theory is identical to the P_1 approximation. If it is necessary to make the diffusion theory more accurate, we can resort to approximations of higher orders, and it is usually understood that this means the P_3 and P_5 approximations, or, in other words, approximations of exclusively uneven orders. So far there has been no mention in scientific literature of any examples of the practical application of the approximations of even orders.

The possibility of using the P_4 approximation is of particular interest since there is reason to believe that it ought to be more accurate than the P_1 approximation (since it is one of the following order) and at the same time less cumbersome than the P_3 approximation. In many cases it could be satisfactorily used as a correction to the diffusion theory, instead of P_3.

The reason why the even approximations are ignored is that

the number of genuine solutions in them, and consequently the number of arbitrary coefficients in the general solution, is smaller than the number of spherical components retained in the expansion of a distribution function. On account of this, it is impossible to satisfy (on the interface of two unlike media) those boundary conditions which are usually imposed in the case of uneven approximations, to wit, conditions of continuity of all these spherical harmonics individually. (It should be pointed out that in non-unidimensional problems this difficulty also arises in the case of uneven approximations.) Below we give a brief description of this method of formulating the boundary conditions, enabling us to use the approximations of any orders, including P_N and the other even ones. For the sake of simplicity we illustrate the method with the example of a plane unidimensional problem. As regards symbols, we are keeping to A. D. Galanin's book /1/.

Let us consider the system of equations for the spherical harmonic method

$$\frac{1-\varkappa_n}{l} f_n + \frac{n}{2n-1} f'_{n-1} + \frac{n+1}{2n+3} f'_{n+1} = 0 \quad (0 \leqslant n < \infty). \tag{1}$$

This system contains an infinite number of equations. The

P_N approximation is that in the equation corresponding to $n=N$, the term $\frac{N+1}{2N+3} f'_{N+1}$ is discarded, and the system then becomes finite. Seeking the solution in a form such that

$$f_n' = -\frac{\alpha}{l} f_n, \tag{2}$$

we reduce the differential equations to normal algebraic ones. From the condition that the determinant of the system is equal to zero (condition of compatibility), we find for α a characteristic equation, the roots of which are Eigenvalues of the problem.

It can be shown that the characteristic equations for different P_N approximations will take the form of polynomials

$$\begin{aligned}
&N=1 \quad \alpha^2 - 3\varepsilon = 0,\\
&N=2 \quad (5+4\varepsilon)\alpha^2 - 15\varepsilon = 0,\\
&N=3 \quad 9\alpha^4 - (55\varepsilon + 35)\alpha^2 + 105\varepsilon = 0,\\
&N=4 \quad (64\varepsilon + 161)\alpha^4 - (735\varepsilon + 315)\alpha^2 + 945\varepsilon = 0.
\end{aligned} \tag{3}$$

Here $\varepsilon = 1 - x = \frac{l}{l_a}$.

This type of equation corresponds to a case of isotropic scattering, when all the c_n, except for $c_0 = 1$, are equal to zero. This, however, is of no importance in principle.

As we see, the degree of the characteristic polynomials with respect to α is equal to $N + 1$, if N is odd, and N, if N is even (this regularity exists as well for $N \geq 4$). Consequently, the number of Eigenvalues of α_j, and therefore the number of partial solutions determined by Eq.(2) at different α_j will always be even, and equal to $N + 1$ or N, according to the parity of the approximation. Thus, the number of arbitrary coefficients in the general solution only coincides with the number of spherical hammonics for uneven N, for in the P_N approximation for a plane unidimensional problem the number of them is equal to $N + 1$.

Eqs.(1) still hold even when their coefficients reflecting the properties of the medium are arbitary functions of the coordinates. We can therefore consider that they have been written down for the medium as a whole, and that in a heterogeneous medium the material characteristics 1, χ and c take the form of piecewise-constant functions.

Let us integrate each equation from the arbitrary lower limit

$x \geq a$, and get as a result

$$\frac{n}{2n-1} f_{n-1} + \frac{n+1}{2n+3} f_{n+1} = \int_a^x \frac{1 - x\varepsilon_n}{l} f dx + \left[\frac{n}{2n-1} f_{n-1} + \frac{n+1}{2n+3} f_{n+1} \right]_{x=a}. \tag{4}$$

There is always a limited function under the integral sign, hence the right hand side of Eq. (4) is always continuous. This implies the requirement of continuity in the expressions

$$J_n \equiv \frac{n}{2n-1} f_{n-1} + \frac{n+1}{2n+3} f_{n+1} \quad (0 < n < \infty). \tag{5}$$

This requirement should be satisfied as well on the interface between two media.

The combinations J_n are coefficients of the serial expansion with respect to the spherical harmonics of the function $f(x, \vartheta)\cos\vartheta$. Indeed,

$$\frac{2n+1}{4\pi} \int f \cos\vartheta P_n(\cos\vartheta)\, d\Omega = \frac{n}{2n-1} f_{n-1} + \frac{n+1}{2n+3} f_{n+1}. \tag{6}$$

Consequently, the system of conditions (5) can be considered as the continuity condition of the function $f(x, \vartheta)\cos\vartheta$. . As we see, conditions (5) are equivalent to the continuity requirement of the integrals

$$\int f \cos\vartheta P_n(\cos\vartheta)\, d\Omega \quad (0 \leqslant n < \infty). \tag{7}$$

At $n \neq 0$ we have a continuity condition for the diffusion flow.

In the P_N approximation $J_N = \frac{N}{2N-1} f_{N-1}$, which implies the continuity f_{N-1}. Taking this fact into account and using J_{N-1}, we arrive at the continuity f_{N-1}, etc. It is not possible to draw a similar conclusion with regard to the functions f_N, f_{N-1}, and so on. They are only part of the continuity condition in the form of combinations (5).

Thus, conditions on the boundary between the two media reduce to the condition of continuity for the following values.

$$\begin{aligned} &f_{N-1};\ f_{N-3};\ f_{N-5} \cdots f_{0(1)};\\ &J_{N-1};\ J_{N-3};\ J_{N-5} \cdots J_{0(1)}. \end{aligned} \tag{8}$$

The minimum value of the subscript in functions (8) depends on the parity of N. The number of joining equations is obviously even in every case. It is not difficult to calculate that for an uneven N this number is $N + 1$, and for an even N it is N, i.e., it is also coincides with the number of arbitary coefficients in the solution.

In particular, for the $P_{\hat{2}}$ approximation we only find two conditions, expressed in the continuity of the functions

$$f_1(x) = f_0(x) + \frac{2}{5} f_2(x).$$

Typical of even approximations is the fact that the scalar flow, which coincides with the function $f_0(x)$ up to an accuracy of the normalizing multiplier, experiences a value at the interface which drops as N increases. When there is no absorption in either medium, the discontinuity disappears.

For uneven approximations in problems in which the number of conditions (8) coincides with the number of spherical harmonics, the conditions derived are identical to the continuity condition for each separate harmonic.

It has been demonstrated in Ref. /2/ that in the general case conditions of type (8) should be derived from continuity on the boundary of the integrals

$$\int f(r\Omega)\cos(\Omega\nu)\, Y_{nm}(\Omega)\, d\Omega \quad (-n \leqslant m \leqslant n,\ 0 \leqslant n \leqslant N), \tag{9}$$

in which ν is normal to the boundary,

$\underline{Y}_{nm}$ are spherical functions of the general type which can be selected , for instance, in the form

$$Y_{n,\pm m} = P_{nm}(\cos\vartheta)\begin{cases}\cos m\varphi \\ \sin m\varphi\end{cases};$$

$\underline{P}_{nm}$ are the Legendre joint polynomials.

(See /3/ and other courses in mathematical analysis for spherical functions in greater detail.)

Conditions (9), the number of which in the general case in equal to $\underline{N}(\underline{N}+1)$, make the problem a closed one, irrespective of the nature of the geometry or the parity of the approximation.

The numerical examples and theoretical arguments show that in these cases in which the use of approximations of a low order is generally possible, the $\underline{P}_2$ approximation can make the $\underline{P}_1$ approximation considerably more accurate. At the same time, the laborious nature of the $\underline{P}_2$ approximation, compared with the $\underline{P}_1$, increases only slightly in most cases of practical interest.

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CRITICAL MASSES OF URANIUM GRAPHITE REACTORS

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Introduction

No detailed information is available so far in scientific literature on the critical masses of uranium graphite reactors. In view of this the need arose to make the relevant calculations for reactors with a wide variety of neutron energy spectra. Some very interesting calculations have been made by Safonov, but they relate to reactors with 100% enrichment with the uranium isotope U^{235} /1/.

This paper deals with the problem of the critical masses of uranium graphite reactors with different degrees of enrichment.

Basic Equations

In most cases the diffusion approximation is sufficient to calculate the critical masses of reactors. It is assumed that the neutrons in the reactor are moderated by elastic and non-elastic scattering. The effect of neutron thermalization is taken into account in the low-neutron energy region.

The multigroup system of equations for the reactor takes the form

$$\left.\begin{aligned} &\nabla D^j \nabla \varphi^j - \Sigma^j \varphi^j = -q^{j-1} - \chi^j Q(\vec{r}) \\ &\nabla D_\tau \nabla \Phi - \Sigma_{c\tau} \Phi = -q^m \\ &q^{j-1} = \sum_{l=1}^{j-1} f^{l-j} \varphi^l \\ &Q(\vec{r}) = \sum_{j=1}^{m} \nu_f^j \Sigma_f^j \varphi^j + \nu_f \Sigma_{f\tau} \Phi + Q_R \end{aligned}\right\}, \qquad (1)$$

in which $$f^{l-j} = \Sigma_s^l \delta_{l,j-1} + \Sigma_{in}^{l-j}, \quad j = 1, 2, \ldots, m.$$

Here $\Sigma_s^l = \frac{\xi \Sigma_s^l}{\Delta u_j}$, Σ_{in}^{l-j}, Σ^j and Σ_f^j are the neutron-spectrum averaged group cross-sections of moderation, non-elastic scattering, and the total removal and fission cross-sections, while $\underline{D}$ is the diffusion coefficient

$$D^j = \frac{1}{3\Sigma_{tr}^j}, \quad \Sigma_{tr}^j = \Sigma_s^j (1 - \bar{\mu}_0) + \Sigma_c^j,$$

in which Σ_s^j is the elastic scattering cross section, and

μ_0 is the mean cosine of the scattering angle.

In the high-energy region, it is advisable to average the

scattering, capture and fission cross-sections with consideration for the fission spectrum, and in the intermediate energy region with consideration for the [redacted] Fermi spectrum*. The constants D_T, Σ_{cT} and Σ_{fT} for the heat group have to be averaged over the steady-state neutron spectrum in an infinite homogeneous medium, taking into account the thermal motion of the moderator nuclei. For this purpose we use the model of a single-atom gas moderator put forward by Koen /2/. The corresponding equation for neutron density N derived in Wilkins' differential form /2/** is

$$\frac{d^2N}{dx^2} = \left(\frac{1}{x} - 2x\right)\frac{dN}{dx} + 4\left(\frac{\beta}{x} - 1\right)N, \tag{2}$$

in which $x = \frac{v}{\sqrt{2kT}}$ (v is the neutron velocity, T is the absolute temperature of the medium, and k is the Boltzmann constant);

$\beta = \sqrt{\frac{T_0}{T}}\frac{\Sigma_{cT}}{(\xi\Sigma_s)_T}$ [$(\xi\Sigma_s)_T$ is the moderating power at E = 0.025 ev].

The function N(x) at x = 0 satisfies the conditions $N(0) \neq 0$, $N'(0)=0$, $N''(0)=$ const. When solving Eq. (2) we find the function N(x), which is then used to average the physical constants:

$$\Sigma_{c\tau} = \Sigma^0_{c\tau}\sqrt{\frac{T_0}{T}}\frac{\int_0^{x_{\tau p}} N(x)\,dx}{\int_0^{x_{\tau p}} xN(x)\,dx};$$

$$\Sigma_{f\tau} = \Sigma^0_{f\tau}\sqrt{\frac{T_0}{T}}\frac{\int_0^{x_{\tau p}} N(x)\,dx}{\int_0^{x_{\tau p}} xN(x)\,dx},$$

in which $x_{\tau p}$ is the boundary separating the thermal diffusion region from the moderation region, and

$\Sigma^0_{c\tau}$, $\Sigma^0_{f\tau}$ is the absorption and fission cross section for thermal neutrons /3/.

Considerable attention should be given to the resonance neutron capture in the intermediate energy region. Twelve resonance levels have been calculated on U^{235} nuclei in the capture cross section and four levels on the U^{238} nuclei. The corresponding probabilities of avoiding the resonance capture for a particular isotope are calculated by the formula

$$\langle \varphi \rangle_n = e^{-\frac{n}{\xi\Sigma_s}\int_{\varepsilon\varphi}\varphi(T)}.$$

Here n is the resonance number, and

ρ_i is the number of nuclei of the absorber per unit of volume.

The effective resonance integral is determined by the formula

$$J^n_{эф}(T) = J^n \frac{\eta(\xi, h)}{\sqrt{1 + h_n}},$$

in which J^n is the total resonance integral at the level with the number n, and

T is the temperature of the medium;

$h_n = \frac{\Sigma^n_{pi}}{\Sigma_{sn}}$ (Σ^n_{pi} is the absorption cross section at maximum resonance;

Σ_{sn} is the potential scattering cross section);

η is a factor taking into account the Doppler broadening of the resonance lines due to the effect of the thermal motion of moderator nuclei;

$\xi_n = \frac{\Gamma_n}{\Delta_n}$ (Γ_n is the resonance width;

$\Delta_n = 2\sqrt{\frac{kT}{M_i} E_n}$).

The function $\eta(\xi, h)$ is turbulated in /4/. Markelov and Tyutereb have put forward a simple interpolation formula for calculating this function

$$\eta(\xi, k)=\begin{cases}1+\sqrt{\beta^2+\beta\gamma+1}-\sqrt{\beta^2-\beta\gamma+1}, & \text{if } 0{,}09\leqslant\xi<1,\\ 1{,}05, & \text{if } \xi>1,\end{cases}$$

in which

$$\beta=k(0{,}15+\xi);$$

$$\gamma=\frac{0{,}086}{\xi}-0{,}228\xi+0{,}282.$$

This formula was the one used in the calculations.

If the group with number $\underline{j}$ contains so-called resonances, we calculate the total probability of avoiding resonance capture in the group $<\varphi'>$. Here the density of the moderation of the neutrons [illegible] in the group $\underline{j}$

$$\varphi'=\sum_{i=1}^{j} {}^{i-j+1}\varphi^{i}$$

has to be multiplied by the factor $<\varphi'>$.

It should also be noted that there is a definite probability of fission at the resonances of U^{235}. If the probability of resonance capture with fission is designated $1/(1+\alpha_{\underline{n}})$, then the probability of radiation capture is equal to $\alpha_{\underline{n}}/(1+\alpha_{\underline{n}})$. The total number of fissions caused by resonance capture of neutrons on U^{235} nuclei takes the form

$$Q_R = \sum_j \nu_j q^j F^j,$$

in which

$$F^j = \sum_n \frac{1 - \langle \varphi \rangle_n}{1 + a_n},$$

and summation is with respect to all the U^{235} resonances reaching the group with number j.

Multigroup System of Constants

The multigroup of physical constants has been compiled on the basis of published data.

In the fission spectrum region a system of constants has been put forward for U^{235} and U^{238} in Ref. /5/. In the region of intermediate energies the system of constants was derived on the basis of experimental data processed by Malyshev /6/. The constants for the heat group are taken from Ref. /3/. The averaged group constants for graphite have been checked by calculating the neutron age in the graphite. The neutron age up to the indium resonance is 318 cm^2, which tallies well with the experimental value. The multigroup system of physical constants is given in Tables 1-3, and the resonance parameters for U^{235} and U^{238} are given in Tables 4 and 5.

TABLE 1

No. of group	Δu	χ	U_{zm}				U_{zm}				C		
			σ	$\nu_f \sigma_f$	σ_{fr}	σ_a	σ	$\nu_f \sigma_f$	σ_{fr}	σ_a	σ	σ_{tr}	σ_a
1	−1,625 + 0,375	0,5674	2,92	3,70	4,3	0,004	2,71	1,62	4,3	0,005	0,353	1,44	0,353
2	+0,875 + 1,000	0,2158	2,55	3,19	4,8	0,030	1,13	0	4,8	0,040	0,632	2,16	0,632
3	1,000 + 1,625	0,1191	1,97	3,16	5,7	0,050	0,53	0	5,8	0,060	0,774	2,87	0,774
4	1,625 + 2,250	0,0559	1,90	3,56	7,4	0,050	0,45	0	7,4	0,060	0,886	3,41	0,886
5	2,250 + 3,000	0,0269	2,23	4,23	9,6	0,064	0,45	0	9,5	0,080	0,948	3,94	0,948
6	3,0 + 3,75	0,0092	2,77	5,21	11,5	0,075	0,49	0	11,3	0,092	0,969	4,33	0,969
7	3,75 + 4,50	0,0031	3,54	6,42	12,8	0,090	0,54	0	12,4	0,080	0,990	4,35	0,990
8	4,50 + 5,25	0,0010	4,97	8,40	14,5	0,115	0,67	0	14,0	0,123	0,990	4,44	0,990
9	5,25 + 7,00	0,0005	7,11	12,3	17,4	0,0494	1,14	0	12,1	0,0528	0,424	4,44	0,424
10	7,00 + 8,50	0;0000	14,9	24,6	23,7	0,0577	4,00	0	14,9	0,0616	0,495	4,44	0,495
11	8,50 + 9,50	0	27,1	44,3	35,4	0,0865	9,21	0	20,1	0,0924	0,743	4,44	0,743
12	9,50 + 10,5	0	39,9	67,4	51,8	0,0865	9,91	0	20,8	0,0924	0,742	4,14	0,742
13	10,5 + 11,5	0	38,6	64,9	101,0	0,0949	0,0924	0	11,0	0,0924	0,742	4,44	0,742
14	11,5 + 12,5	0	24,1	39,4	112,0	0,0949	0,0924	0	11,0	0,0924	0,742	4,44	0,742
15	12,5 + 13,5	0	26,6	45,8	52,9	0,105	0,279	0	11,2	0,0924	0,742	4,44	0,742
16	13,5 + 14,5	0	39,2	76,0	48,9	0,118	0,433	0	11,4	0,0924	0,742	4,44	0,742
17	14,5 + 15,5	0	88,6	191,0	96,1	0,134	0,655	0	11,5	0,0924	0,742	4,44	0,742
for heat group			$\sigma_{cr} = 687$; $\sigma_{fr} = 580$; $\sigma_a(1-\bar{\mu}_0) = 9,97$; $\sigma_{cr} = 2,75$; $\sigma_a(1-\bar{\mu}_0) = 9,972$;								$\sigma_{cr} = 0,0032$; $\sigma_a(1-\bar{\mu}_0) = 4,53$		

TABLE 2

Matrix of inelastic scattering U^{236} $(\nu_{U^{236}})^{l-l'}_{l'}$

l'	l = 1	2	3	4	5	6
2	0,50	—	—	—	—	—
3	0,49	0,38	—	—	—	—
4	0,35	0,45	0,36	—	—	—
5	0,14	0,22	0,10	0,14	—	—
6	0,05	0,09	0,03	0,03	0,076	—
7	0,02	0,03	0,01	0,01	0,01	0,025

TABLE 3

Matrix of inelastic scattering U^{236} $(\nu_{U^{236}})^{l-l'}_{l'}$

l'	l = 1	2	3	4	5	6
2	0,71	—	—	—	—	—
3	0,60	0,40	—	—	—	—
4	0,50	0,35	0,34	—	—	—
5	0,20	0,14	0	0,24	—	—
6	0,07	0,05	0	0	0,15	—
7	0,02	0,02	0	0	0	0,088

TABLE 4

Matrix of inelastic scattering U^{236}

No. of resonance	n	E	σ_r	J	ξ	a
1	11	35,3	1730	9,35	0,987	0,4
2	11	34,6	1510	5,76	0,692	0,69
3	11	33,7	800	3,72	0,835	0,50
4	11	32,2	700	3,40	0,854	0,50
5	11,25	23,6	900	6,00	0,998	0,50
6	11,50	21,1	700	5,05	1,06	0,50
7	11,50	19,40	1590	15,4	1,32	0,62
8	12,00	12,40	2340	18,4	0,854	1,71
9	12,00	11,65	1840	10,5	0,996	5,85
10	12,25	8,82	1970	33,8	1,57	0,67
11	12,75	6,39	1120	14,5	1,02	0,96
12	13,00	4,84	530	5,0	0,639	6,25

Table 5

Resonance Parameters for U^{238}

№ of resonances	a	E	σ_r	J	ξ
1	10,25	66,3	21190	10,03	0,258
2	11	36,8	39·20	44,24	0,475
3	11,50	21,0	33030	61,84	0,362
4	12,50	6,68	22030	129,0	0,500

Calculation of Reflectorless Reactors

The multigroup system of equations for reactors without reflectors takes the form

$$\left.\begin{aligned} &(\varkappa^2 D^j + \Sigma_c^j)\,\varphi^j = q^j + \varkappa^j Q \\ &(\varkappa^2 D_\tau + \Sigma_{c\tau})\,\Phi = q_\tau \\ &q^j = \sum_{i=1}^{j-1} f^{ij}_{\alpha\beta}\,\varphi^i \\ &Q = \sum_{j=1}^{m} \nu_f^j \Sigma_{f\tau}^j \varphi^j + \nu_f \Sigma_{f\tau} \Phi \end{aligned}\right\} \qquad (3)$$

$$\Sigma^{l-1}_{f\,\text{эфф}} = \begin{cases} \Sigma^{l-1}_f \langle \varphi \rangle^{l-1} \\ \Sigma^{l-1}_f \end{cases} \qquad (3)$$

Here in groups in which there are resonances, and

$$\Sigma^{l}_{f\,\text{эфф}} = \Sigma^{l}_f + \left(\frac{\xi\Sigma_s}{A_M}\right)^{l} F^{l};$$

in the remaining groups;

χ^2 is the geometrical parameter and the first Eigenvalue of the problem

$$\left.\begin{aligned} \nabla^2\psi + \chi^2\psi = 0 \\ \psi = 0 \text{ на } S_э \end{aligned}\right\} \qquad (4)$$

($S_э$ is the extrapolated reactor surface).

It is assumed that in the resonance energy region the moderation density q is expressed in terms of the flux φ by the formula

$$q = \xi \Sigma_s \varphi.$$

The problem is solved by the method of successive approximations, which is normally called the method of source integration /7/.

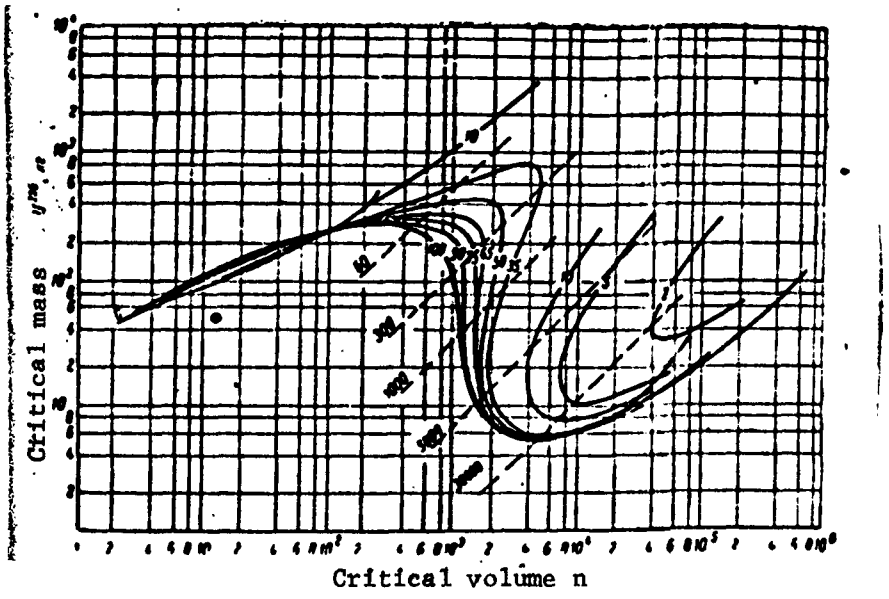

Fig. 1. Critical masses of spherical uranium-graphite reactors without reflectors (y_C = 1.65 g/cm^3)
Broken lines join points with identical a;
Solid lines show different degrees of enrichment with isotope U^{235} (%).

We should point out that during transition from an extrapolated surface to a true surface, the extrapolated length was averaged over the energy neutron spectrum in the reactor. Solution of the problem gives the critical loads and critical volumes of the spherical reactors without reflectors for a wide range of the ratio $a = \frac{\rho_c}{\rho_{U^{235}}}$ at different degrees of enrichment of the uranium by isotope U^{235}.

The results of calculation of the critical masses of the reactors are given in Fig. 1.

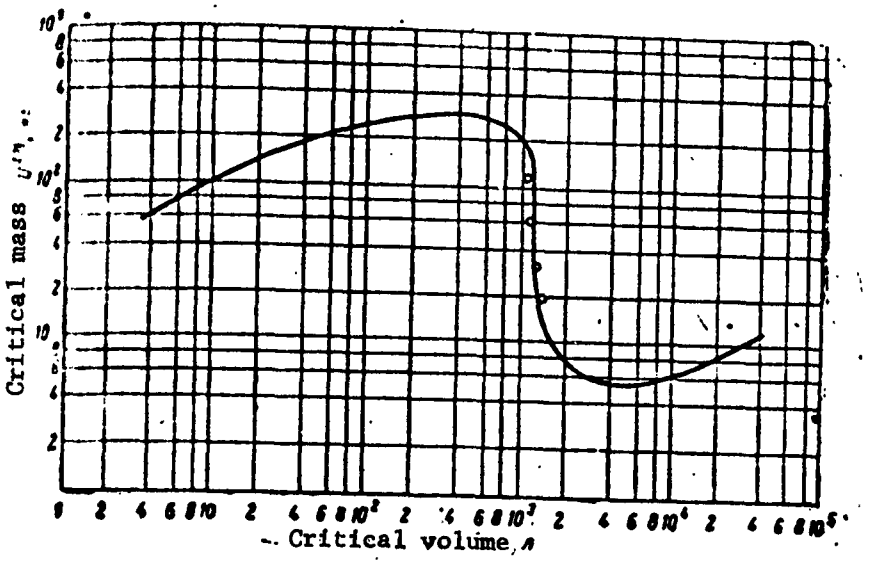

Fig. 2. Critical masses of uranium-graphite reactors without reflectors as function of volume.

Solid line - theoretical data , uranium enrichment 90%. y_C = 1.65 g/cm^3

o - experimental data, uranium enrichment 93.2% y_C = 1.645 g/cm^3

Fig. 2 compares the results of the calculations and measurements made in uranium-graphite reactors without reflectors /8/. The close correspondence of the experimental and theoretical data is observed in the region of both thermal neutron reactors and intermediate neutron reactors.

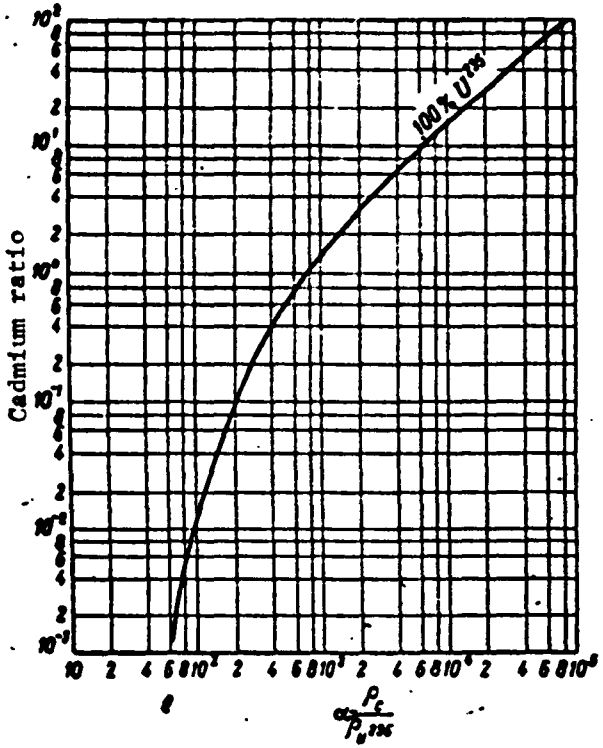

Fig. 3. Cadmium ratio as function of $\alpha = \rho_c / \rho_{U^{235}}$.

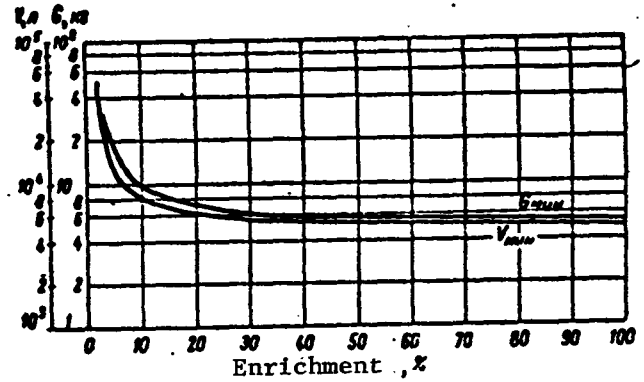

Fig. 4. Minimum critical masses and their volumes for uranium-graphite reactors without reflectors.

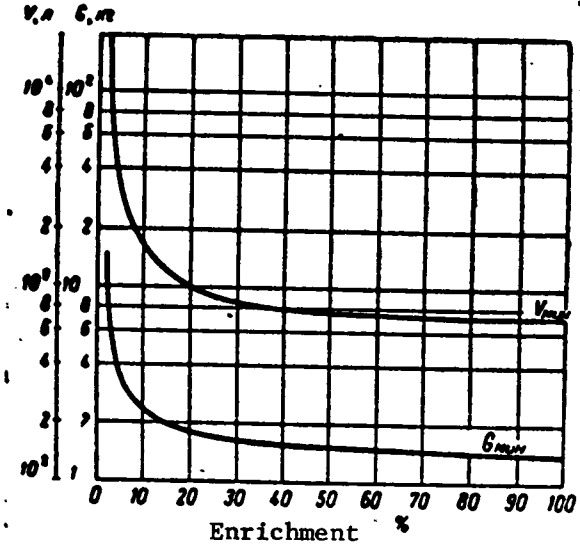

Fig. 5. Minimum critical masses and their volumes for uranium - graphite reactors with reflectors

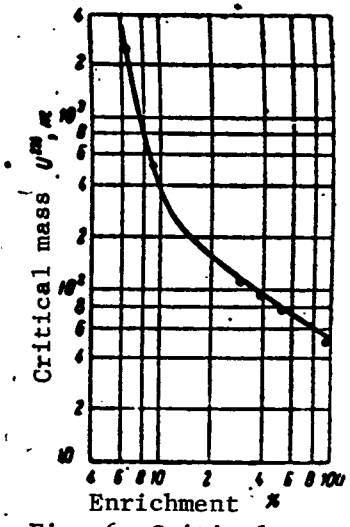

Fig. 6. Critical masses of uranium metal reactors with different enrichment and without reflectors.

Solid line – theoretical data;
•— experimental data

Fig. 3 shows the cadmium ratio for the fission chamber of the function of α . The curve shown corresponds to a 100% uranium enrichment. The curves for other uranium enrichments coincide for practical purposes with the ones in shown Fig. 3.

Figs. 4 and 5 show graphs for minimum critical masses and their volumes. Fig. 6 compares the results of calculation of critical masses with experimental data for pure uranium as a function of enrichment /9/. Analysis that the diffusion approximation leads to substantial errors, even for small systems.

Calculation of Spherical Reactors With Reflectors

Reactors with graphite reflectors used to be calculated by the multigroup method, the principle of which is described in Ref. /7/. It was assumed that the reflector was of infinite thickness, but that the calculation was made with a reflector thickness of 70 cm. The error due to substitution of a finite for an infinite reflector was comparable with the errors of the numerical calculation

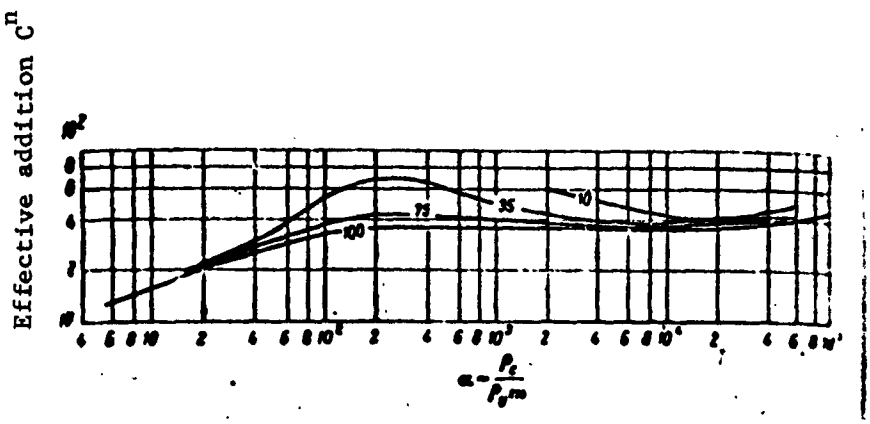

Fig. 7. Economic advantage of graphite reflector for spherical uranium-graphite reactors Ye = 1,65 g/cm^3
Figures on curves show enrichment with isotope U^{235} (%)

Fig. 7 gives the effective additions ~~of~~ as a function of α at different degrees of enrichment, derived ~~during~~ from numerical solution of multigroup reactor equations. The maximum value of the effective addition for reactors with a 100% uranium enrichment is found in thermal reactors and is equal to the square root of the migration area $\sqrt{M}R$ = 64 cm, and decreases monotonically to 8 cm for fast reactors. For intermediate neutron reactors we observe an increase in the effective addition.

Given other degrees of uranium enrichment, the curve showing the effective addition of a function of α is also complex in form. When the enrichment decreases, the maximum effect of addition is attained in the region of intermediate neutron spectrum reactors.

It may be assumed that there is a region in which infinitely large subcritical reactors without reflectors may become critical if reflectors are used. This means that an infinitely multiplying medium coinciding in properties with the active zone of the reactor is subcritical ($k_{eff} = k_{\infty} < 1$). At the same time, if the size of the active zone surrounding the reflector is restricted, the system may become supercritical.

It should be pointed out that to calculate fast neutron reactors, the diffusion approximation is not sufficiently accurate. Fig. 8 gives the critical masses of spherical uranium-graphite reactors with infinite graphite reflectors.

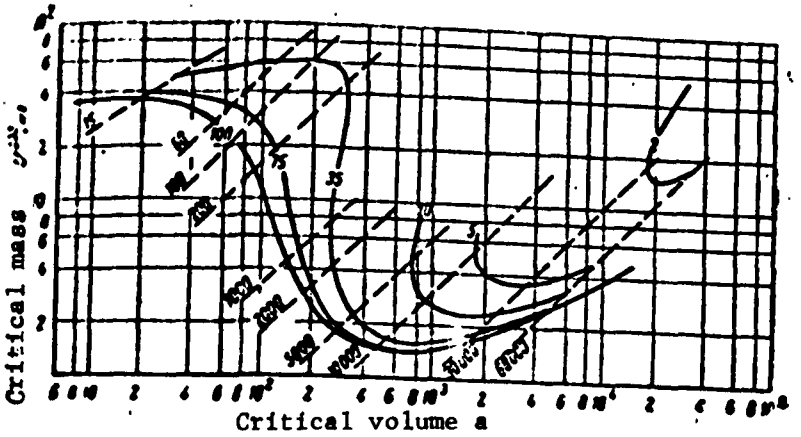

Fig. 8. The critical masses of spherical uranium-graphite reactors with infinite graphite reflectors. Broken lines join points with identical a. solid lines show enrichment with isotope U^{235} (%).

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CRITICAL MASSES OF URANIUM-BERYLLIUM REACTORS

by

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A number of papers /1/ deal with the calculation of critical masses of uranium -beryllium reactors, but they only touch on calculations of reactors without reflectors with 100% uranium enrichment with U^{235}. The present paper contains the results of calculating critical masses of uranium-beryllium reactors both without a reflector and with an infinite beryllium reflector at different degrees of enrichment with isotope U^{235}. All the calculations have been made in the diffusion-age approximation by the multigroup method set forth in the paper by Marchuk and others entitled "Critical masses of Uranium-graphite Reactors", in which uranium-graphite systems are investigated in a similar fashion.

Group constants

Letargy	σ^{Be}	σ^{Be}_{tr}	σ^{Be}_{s}	$\frac{l \to l}{f_{Be}}$
— 1,625 — 0,375	0,44	1,5	0,050	0,35
0,375 — 1,0	0,65	2,6	0,650	0,08
1,0 — 1,625	1,00	3,4	1,000	0,02
1,625 — 2,25	0,90	3,9	0,900	—
2,25 — 3,0	1,10	4,8	1,100	—
3,0 — 3,75	1,30	5,4	1,300	—
3,75 — 4,5	1,3	5,5	1,300	—
4,5 — 5,25	1,66	5,60	1,66	—
5,25 — 7,0	0,710	5,56	0,710	—
7,0 — 8,5	0,828	5,56	0,828	—
8,5 — 9,5	1,24	5,56	1,24	—
9,5 — 10,5	1,24	5,56	1,24	—
10,5 — 11,5	1,24	5,56	1,24	—
11,5 — 12,5	1,24	5,56	1,24	—
12,5 — 13,5	1,24	5,56	1,24	—
13,5 — 14,5	1,24	5,58	1,24	—
14,5 — 15,5	1,20	5,46	1,20	—

Note: for the heat group $\sigma^{Be}_{ст} = 0{,}01$; $\sigma^{Be}_{st}(1 - \cos\vartheta)^{Be} = 6{,}48$; $\xi\sigma^{Be}_{ст} = 1{,}46$.

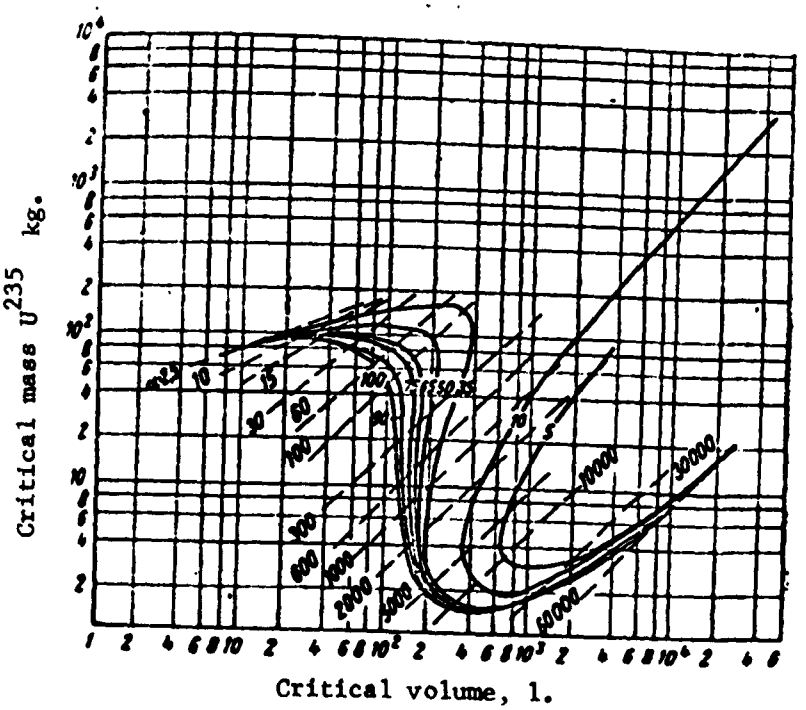

Fig. 1. Critical masses of spherical uranium-beryllium reactors without reflectors ($\gamma_{Be} = 1{,}65$ g/cm^3); the dashed lines show join points with the same α; the solid lines correspond to different enrichments U^{25} (in %)

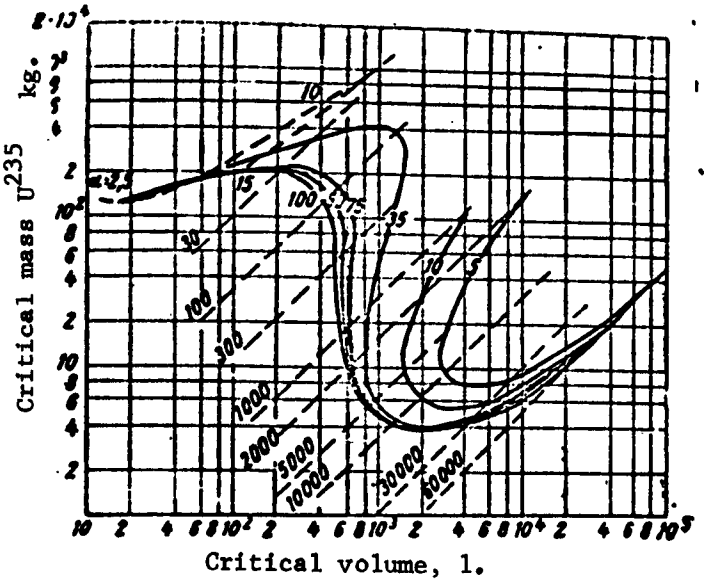

Fig. 2. Critical masses of spherical uranium-beryllium reactors without reflectors (γ_{Be} = 1.15 g/cm^3). The meaning of the lines is as in Fig. 1.

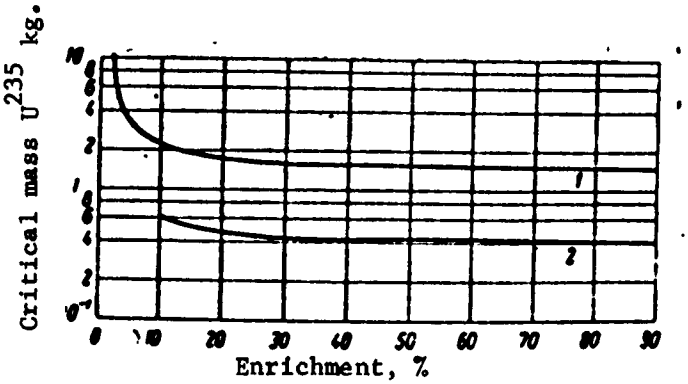

Fig. 3. Minimal critical masses of uranium beryllium reactors (γ_{Be} = 1.85 g/cm^3).

1) without reflector; 2) with beryllium reflector.

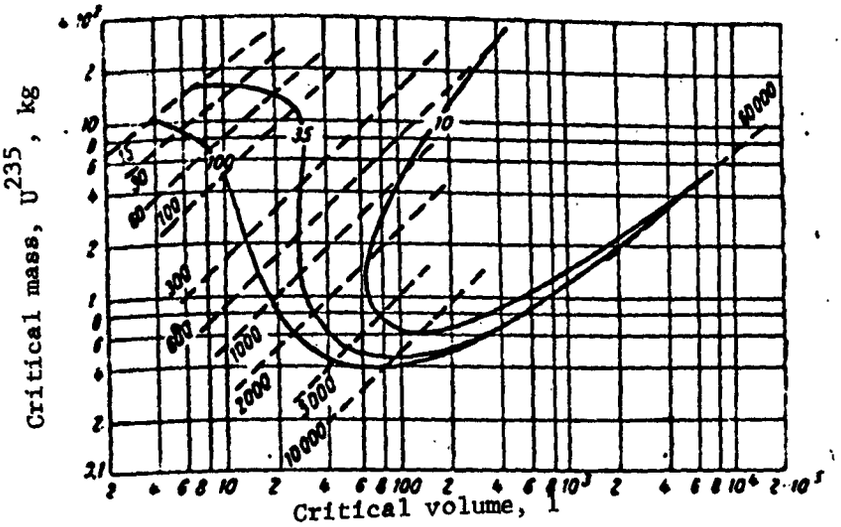

Fig. 4. Critical masses of uranium-beryllium reactors with beryllium reflector (γ_{Be} = 1.85 g/cm^3). Meaning of lines as in previous figures.

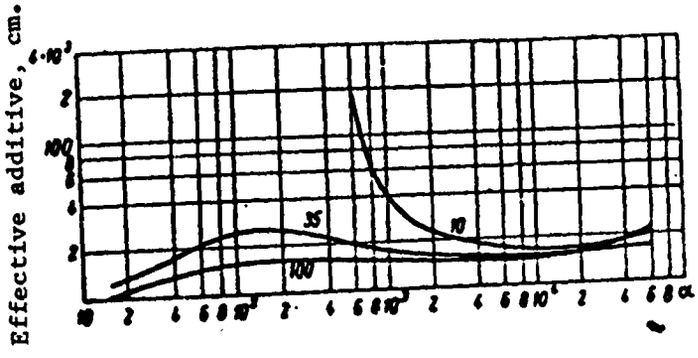

Fig. 5. Effective additives for reactors without reflectors as function of

for three enrichments (γ_{Be} = 1.85 g/cm^3).

In the high energy region, apart from elastic scattering, we calculated nonelastic scattering in uranium and the reaction (n, 2n) on beryllium. The corresponding group constants are given in Refs. /3/. To obtain the group constants in the intermediate energy region we used data from Ref. /3/. Here the calculations took into account the resonance absorption on U^{235} and U^{238}, and at low energies took into account the thermalization of neutrons within the framework of the model single-atom gas moderator /4/. The heat constants for beryllium are taken from /5/.

The multigroup system of constants for beryllium is given in the Table. The age of the neutrons up to the indium resonance energy, τ_{in}, calculated using these group constants, is 81 cm^4 (beryllium density γ_{Be} = 1.85 g/cm^3), and this tallies well with the experiment.

The results of the calculations of uranium-beryllium reactors without reflectors at ratios $\alpha = \rho_{Be}/\rho_{U^{235}}$ (in which ρ is the nuclear density) and different degrees of enrichment with isotope U^{235} are given in Figs. 1 to 3. The critical load curve corresponding to a 100% enrichment tallies well, qualitatively speaking, with the data in /1/.

When calculating reactors with a reflector, the thickness of

the latter was taken as 49 cm. A beryllium reflector of this thickness can be considered infinite. Results of the calculations of the critical masses of U^{235} and the effective additions are given in Figs. 4 and 5 respectively.

Calculations were made with a Strela (arrow) computer.

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CRITICAL MASSES OF WATER MIXTURES OF URANIUM AND PLUTONIUM COMPOUNDS

by

G. I. Marchuk, G. A. Ilyasova, V. Ye. Kolesov, V. P. Kochergin, L. I. Kuznetsova and Ye. I. Pogudalina.

Introduction

Knowledge of the critical masses of different uranium and plutonium compounds is very important both in designing and building nuclear reactors as well as in solving problems involving safety. Systems with a water-containing moderator are of great practical interest. The results of some experimental investigations of critical masses of uranium-water mixtures have recently been published. In particular, Ref. /1/ discusses a large number of experiments made by scientists.

It is an extremely difficult thing to calculate the critical masses of uranium-water and plutonium-water reactors. The main difficulty is that hydrogen moderation cannot be considered continuous, and on that account the age theory of calculation proves inapplicable. So far comparatively little theoretical data on the critical masses of water-containing systems has been published. We can therefore only refer to Sasonov's calculations which are

given in /2/.

The authors made a large number of calculations of the critical masses of reactors both without reflectors and with water reflectors. The fuel used was UO_2 with water and a mixture of UO_2 + PuO_2 with water. The concentration of the fissile matter and the mixture varied within wide limits. The calculations were made for different degrees of enrichment and different ratios of the number of uranium nuclei to plutonium. The error in the critical mass did not exceed 30%.

All the calculations were made with a Strela computer designed by the Academy of Sciences Computing Centre.

Basic Equations

The multigroup system of reactor equations in the diffusion approximation in the presence of hydrogen can be as follows

$$
\begin{aligned}
&\nabla\varphi_1^j + \Sigma^j \varphi_0^j = q^{j-1} + \chi^j Q(\vec{r}),\\
&\frac{1}{3}\nabla\varphi_0^j + \Sigma_{tr}^j \varphi_1^j = \sum_{l=1}^{j-1} \sum_{s1}^{l-j} \varphi_1^l,\\
&\nabla\Phi_1 + \Sigma_{c\tau}\Phi_0 = q^m,\\
&\frac{1}{3}\nabla\Phi_0 + \Sigma_{tr\tau}\Phi_1 = \sum_{l=1}^{m} \sum_{s1}^{l-m+1} \varphi_1^l,\\
&Q(\vec{r}) = \sum_{j=1}^{m} \nu_f^j \Sigma_f^j \varphi_0^j + \nu_{f\tau}\Sigma_{f\tau}\Phi_0 + Q_R,\\
&q^j = \sum_{l=1}^{j} \left(\sum_{ln}^{l-j+1} + \sum_{s0}^{l-j+1} \right) \varphi_0^l + \Sigma_s^j \varphi_0^j \quad (j = 1, 2, \ldots m)
\end{aligned}
\tag{1}
$$

Here $\sum_{ln}^{l-j}$ are the group nonelastic scattering cross sections from the group with number l into group with the number j;

$\Sigma_s^j = \sum_\nu \left(\frac{\xi\Sigma_s}{\Delta u}\right)_\nu^j$ are the group moderation cross sections on all nuclei, except hydrogen,

$\sum_{s0}^{l-j}$ and $\sum_{s1}^{l-j}$ are values describing the elastic scattering on hydrogen nuclei from group l into group j;

is the total cross section of removal from group j.

The heat constants were averaged by the formuli

$$
\Sigma_{c\tau} = \Sigma_{c\tau}^0 \sqrt{\frac{T_0}{T}} \frac{\int_0^{x_{\tau\rho}} N(x)\,dx}{\int_0^{x_{\tau\rho}} xN(x)\,dx},
$$

$$\Sigma_{fr} = \Sigma^0_{fr} \sqrt{\frac{T_0}{T}} \frac{\int_0^{x_{tp}} N(x)\,dx}{\int_0^{x_{tp}} xN(x)\,dx},$$

$$\Sigma_{tr} = \Sigma_{st}(1 - \bar{\mu}_{0r}) + \Sigma_{cr},$$

in which Σ'^0_{cr}, Σ^0_{fr}, $\Sigma_{sr}(1 - \bar{\mu}_{0r})$ are the thermal cross sections at an energy 0.025 ev,

x_{tp} is the boundary separating the thermal diffusion region from the moderation region,

T is the temperature of the medium,

T_0 is the room temperature.

The function $N(x)$ is the neutron spectrum in an infinite homogeneous medium, allowing for the thermal motion of the moderator nuclei.

To find N(x) we use ~~the equation~~ Eq. (3)

$$y''(x) = S(x)y'(x) - R(x)y(x)$$

with the initial conditions

$$y(0) = 0, \qquad y'(0) = \text{const.}$$

Here

$$S(x) = P(x)\sqrt{\pi}\,\mathrm{erf}(x);$$

$$R(x) = P(x)\,e^{-x^2} - x^2 + r(x)\frac{4}{\sqrt{\pi}P(x)};$$

$$P(x) = \frac{1}{\exp[-x^2] + \sqrt{\pi}x\,\mathrm{erf}(x)};$$

$$r(x) = \frac{\Sigma_{sr}^{H}}{\Sigma_{cr}^{0} + \left[\frac{1}{2x}\,\mathrm{erf}(x) + \frac{1}{\sqrt{\pi}P(x)}\right]\Sigma_{sr}^{H}},$$

$$N(x) = r(x)\,y(x)\,x e^{-\frac{x^2}{2}}.$$

and

The resonances in the absorption cross sections were taken into account in the following way. In the groups with resonances the moderation density q was multiplied by the probability of avoiding resonance capture $\langle\varphi'\rangle$. The calculation of $\langle\varphi'\rangle$ was done in the same way as described in the article "Critical masses of uranium-graphite reactors" by Marchuk and others in this collection. Twelve

resonances on U^{235} and four resonances on U^{238} were calculated.

The number of neutrons obtained during fission through resonance absorption on isotope U^{235} is equal to

$$Q_R = \sum_j \nu_j q' F_j',$$

in which

$$F_j' = \sum_n \frac{1 - \langle \varphi \rangle_n}{1 + \alpha_n},$$

and summation is over all the resonances of U^{235} which are found in the group with the number j.

When calculating reactors without reflectors, system (1) was solved by separation of the variables. In the case of bizonal equipment, the method of source integration was used to find the eigenvalue of the problem, while the system of equations (1) was solved by the finite-difference factorization method /4/.

Table 9

Multigroup system of constants for Pu, H and O

l	Δu	x	Pu^{239}				H		O		
			σ_a	$\nu\sigma_f$	σ_{tr}	σ_s	σ_a	σ_{tr}	σ_a	σ_{tr}	σ_s
1	1,625 + 0,375	0,56741	3,0	6,277	4,4	0,003	1,50	1,55	0,200	1,21	0,200
2	0,375 + 1,000	0,21578	2,84	5,287	5,1	0,030	3,32	3,39	0,550	3,16	0,550
3	1,000 + 1,625	0,11911	2,25	4,802	6,2	0,050	4,70	4,75	0,740	3,42	0,740
4	1,625 + 2,25	0,05594	2,03	4,768	7,6	0,050	6,45	6,63	0,630	3,64	0,630
5	2,25 + 3,00	0,02639	2,10	4,781	9,8	0,060	8,14	8,46	0,580	3,35	0,580
6	3,00 + 3,75	0,00917	2,36	5,166	11,5	0,090	10,3	10,7	0,580	3,35	0,580
7	3,75 + 4,50	0,00308	2,83	5,863	12,8	0,100	12,2	12,7	0,390	3,35	0,390
8	4,50 + 5,25	0,00102	3,42	6,578	15,0	0,1068	13,7	14,3	0,613	3,35	0,612
9	5,25 + 7,00	0,00046	5,26	12,68	18,2	0,0457	9,49	11,4	0,263	3,64	0,263
10	7,00 + 8,5	0,00003	11,8	21,46	14,0	0,0533	10,4	12,0	0,306	3,64	0,306
11	8,5 + 9,5	0	30,5	53,79	33,1	0,0801	12,7	13,7	0,459	3,64	0,459
12	9,5 + 10,5	0	63,4	105,6	81,3	0,0801	12,7	13,7	0,459	3,64	0,459
13	10,5 + 11,5	0	89,4	131,4	78,0	0,0801	12,7	13,7	0,459	3,64	0,459
14	11,5 + 12,5	0	129,0	237,9	91,5	0,0801	12,7	13,7	0,459	3,64	0,459
15	12,5 + 13,5	0	44,7	68,27	34,8	0,0801	12,8	13,7	0,459	3,64	0,459
16	13,5 + 14,5	0	24,6	58,14	34,6	0,0801	13,1	14,1	0,459	3,64	0,459
17	14,5 + 15,5	0	454,0	791,5	473,0	0,0801	14,0	15,0	0,459	3,64	0,459
	ℓ										

Table 2.

Group cross sections for elastic scattering on hydrogen

$l \to l + k$

σ_{ln}

l	k						
	1	2	3	4	5	6	7
1	0,6964	0,3728	0,1995	0,1212	0,0573	0,0271	0,0242
2	1,5427	0,8257	0,5018	0,2370	0,1120	0,0529	0,0473
3	2,1823	1,3262	0,6264	0,2959	0,1398	0,1034	0,0217
4	3,4020	1,6059	0,7591	0,3586	0,2652	0,0434	0,0124
5	4,2960	2,0294	0,9585	0,7090	0,1159	0,0210	0,0122
6	5,4497	2,5741	1,9041	0,3111	0,0555	0,0208	0,0121
7	6,4428	4,7658	0,7787	0,1414	0,0520	0,0191	0,0111
8	11,294	1,8454	0,3351	0,1233	0,0453	0,0167	0,0097
9	7,3740	1,3388	0,4925	0,1812	0,0667	0,0245	0,0143
10	6,5820	2,4214	0,8907	0,3277	0,1205	0,0443	0,0258
11	8,0324	2,9550	1,0871	0,3999	0,1471	0,0541	0,0315
12	8,0326	2,9548	1,0872	0,3999	0,1473	0,0556	—
13	8,0328	2,9554	1,0870	0,3999	0,2328	—	—
14	8,0332	2,9548	1,0871	0,6327	—	—	—
15	8,0480	2,9609	1,7232	—	—	—	—
16	8,2617	4,8081	—	—	—	—	—
17	13,911	—	—	—	—	—	—

$l \to l + k$

σ_{ln}

l	k						
	1	2	3	4	5	6	7
1	0,4564	0,1787	0,0700	0,0304	0,0099	0,0032	0,0015
2	1,1643	0,4561	0,1982	0 0643	0,0209	0,0068	0,0033
3	1,6188	0,7037	0,2284	0,0741	0,0241	0,0107	0,0005
4	2,5552	0,8299	0,2694	0,0875	0,0390	0,0027	0,0003
5	3,1616	1,0265	0,3332	0,1456	0,0104	0,0009	0,0003
6	4,0001	1,2957	0,5791	0,0405	0,0037	0,0008	0,0002
7	4,7169	2,1032	0,1469	0,0134	0,0030	0,0007	0,0002
8	7,2522	0,5088	0,0466	0,0104	0,0023	0,0005	0,0001
9	4,2640	0,3903	0,0671	0,0194	0,0043	0,0010	0,0003
10	4,1664	0,9296	0,2074	0,04629	0,0103	0,0023	0,0007
11	5,4274	1,2111	0,2702	0,0603	0,0134	0,0030	0,0009
12	5,4274	1,2110	0,2702	0,0603	0,0134	0,0039	—
13	5,4268	1,2109	0,2702	0,0603	0,0173	—	—
14	5,4272	1,2110	0,2702	0,0776	—	—	—
15	5,4404	1,2139	0,3487	—	—	—	—
16	5,5912	1,6059	—	—	—	—	—
17	7,6771	—	—	—	—	—	—

TABLE 3

Group cross sections of non-elastic scattering on plutonium

$\sigma_{in_{Pu^{239}}}$

	l					
	1	2	3	4	5	6
2	0,32	—	—	—	—	—
3	0,32	0,32	—	—	—	—
4	0,23	0,33	0,28	—	—	—
5	0,09	0,18	0,12	0,15	—	—
6	0,03	0,07	0,04	0,03	0,13	—
7	0,01	0,02	0,01	0,01	0,01	0,04

TABLE 4

System of heat constants

element	σ_{tr}	σ_{fr}	ν_{fr}	$\sigma_{tr}(1-\bar{\mu}_{tr})$
Pu^{239}	1026	746	2,92	9,6
H	0,33	—	—	12,7
O	0	—	—	4,03

The multigroup system of constants for plutonium, oxygen and hydrogen in given in Tables 1-4. The corresponding multigroup constants for U^{235} and U^{238} are taken fo be the same as in the paper by Marchuk mentioned above. The multigroup water constants used in the calculation give a value of the for moderation length square down to indium resonance, equal to 26.5 cm^2, $\overline{r^4}$ =0.665 $\cdot 10^5$ cm^4 (in the diffusion approximation); r_{exc} is 1.222 $\cdot 10^5$ cm^4.

Results of Calculations

The above described method was used to find the critical masses of spherical reactors with water as the moderator. Both reactors without reflectors as well as those with a water reflector 24.5 cm thick were calculated. A reflector of this kind is for practical purposes infinite.

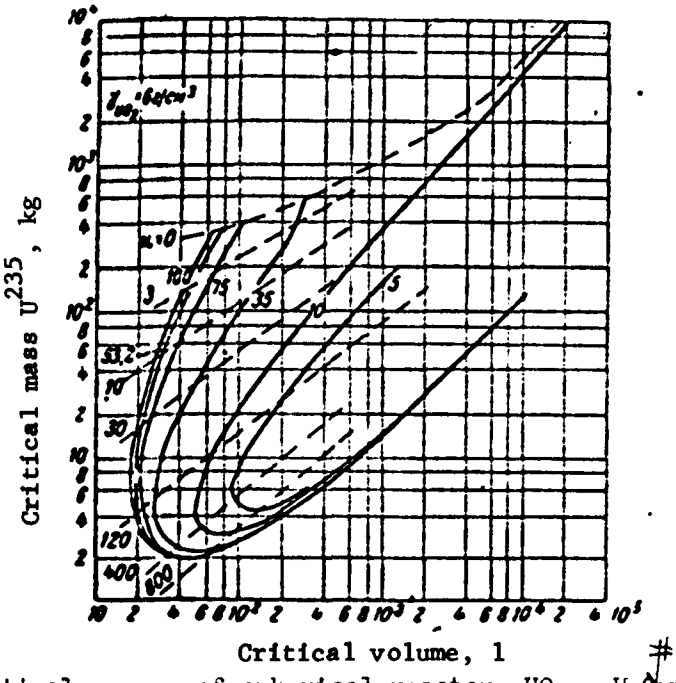

Fig. 1. Critical masses of spherical reactors UO_2 - H_2O without reflectors

Figs. 1-7 show the results of calculations for reactors, the active zone in which is a mixture of UO_2 (γ = 6 g/cm^3) with water. The dependence of the critical masses of the reflectorless reactors on critical volumes at different degrees of enrichment

is shown in Fig. 1. The points with the same ratio of the number of nuclei $x = \rho_H / \rho_{U^{235}}$ are joined by dotted lines. Fig. 2 shows the variation in critical mass as the concentration of the fuel in the mixture changes. The corresponding dependences for reactors with a reflector are given in Figs. 3 and 4. Comparison of the minimum critical masses of reactors with and without a reflector are given in Figure 5. The difference between the extrapolated radius of the reactor without a reflector and the radius of the active zone of the corresponding reactor with a reflector is given Fig. 6. For reflectorless reactors, furthermore, we have calculated the cadmium ratio in the fission chamber

$$CdR = \frac{\int_{u-15.5}^{\infty} \nu_f \Sigma_f \varphi_0 \, du}{\int_{-\infty}^{u-15.5} \nu_f \Sigma_f \varphi_0 \, du}.$$

Fig. 7 shows the CdR as a function of α. It corresponds to a 100% enrichment with U^{235}.

Similar calculations were also made for the mixture UO_2-PuO_2 and water (natural uranium and $UO_{1/2}$, $PuO_{1/2}$, $\gamma_{UO_2} = \gamma_{PuO_2}$ 6 g/cm^3).

The results are given in the form of graphs in Figs. 8-14.

The calculations were made for different ratios $\beta = \rho_U/\rho_{Pu}$.

Fig. 16 shows a comparison of the results of the calculation of critical masses and the measurements published in /1/. The dependence of the extrapolation length $d = 0.71 \frac{\int \lambda_{tr} \varphi_0 du}{\bar{\lambda} \int \varphi_0}$ on fuel concentration in the mixture is shown in Fig. 16.

The value 12.7 given in Table 4 is the transport cross section for single-atom hydrogen. This gave us a certain exaggeration of the diffusion length in water, which was offset by two low a value for the neutron age ($\tau_{theor} = 26.5$ cm^2; $\tau_{exp} = 30$ cm^2). But in accordance with the new experimental value of the neutron age in water (from a report given by A. Veynberg in the Institute of Atomic Energy of the Academy of Sciences of the USSR, $\tau \approx 26$ cm^2), the transport cross section has to be varied in accordance with the experimental value of the diffusion length. Nevertheless, errrors in the critical mass due to this effect are only slight over a wide range of ratios

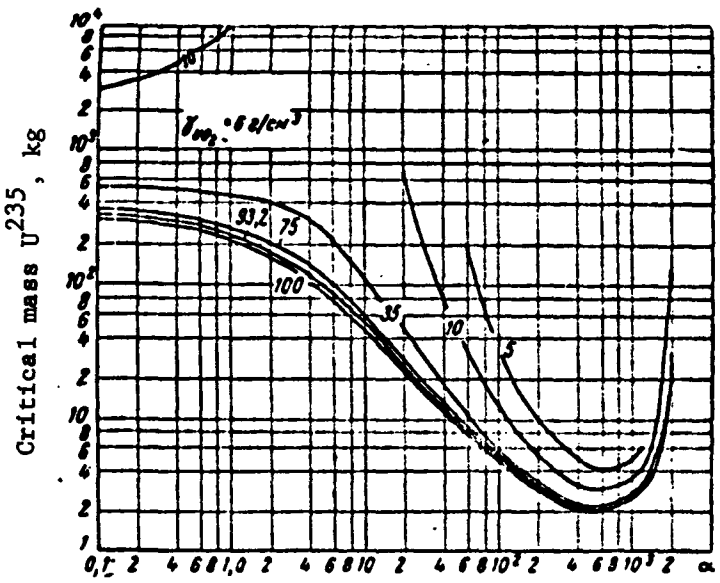

Fig. 2. Critical masses of spherical reactors UO_2 - H_2O without reflectors as function of $\alpha = N_H/N_{U^{235}}$. The figures on the curves correspond to enrichment with U (in %)

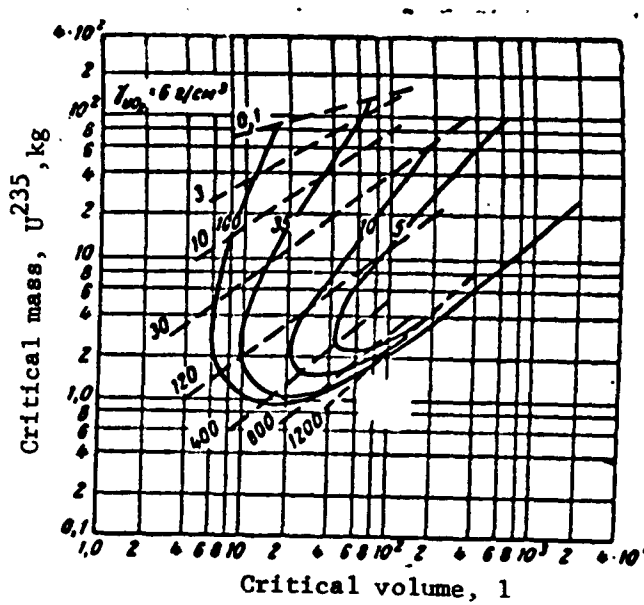

Fig. 3. Critical masses of spherical reactors UO_2 - H_2O with water reflector.

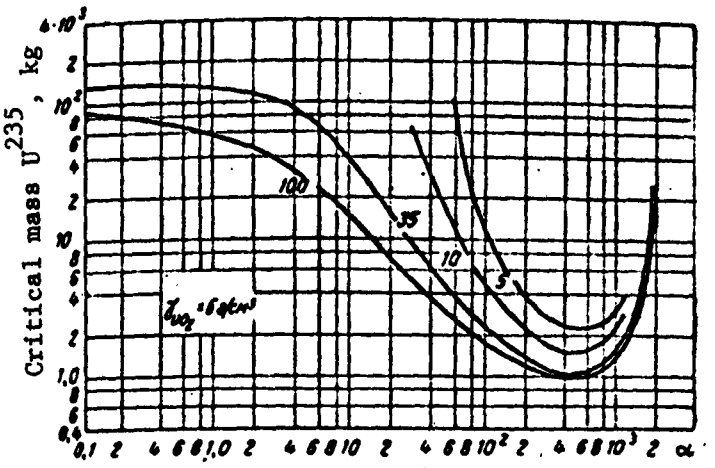

Fig. 4. Critical masses of spherical reactors UO_2 - H_2O with water reflector as function of $\alpha \simeq P_H / P_{U^{235}}$

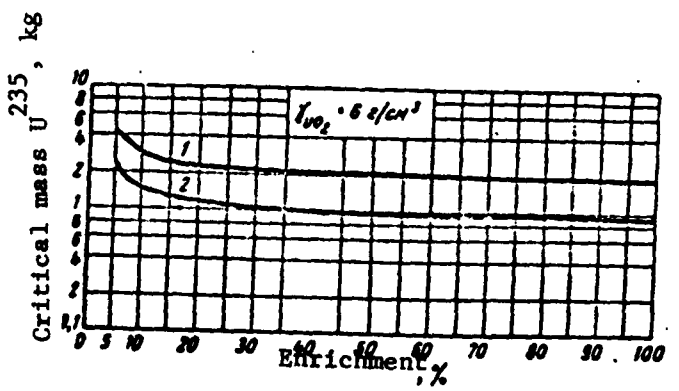

Fig. 5. Minimal critical masses of reactors UO_2 - H_2O:

1) without reflector; 2) with water reflector

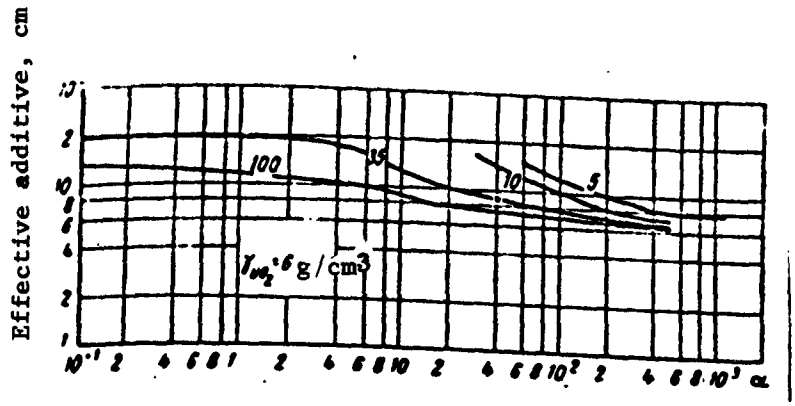

Fig. 6. Economy of water reflector for spherical reactors UO_2 - H_2O ($\alpha = \rho_H/\rho_{U^{235}}$).

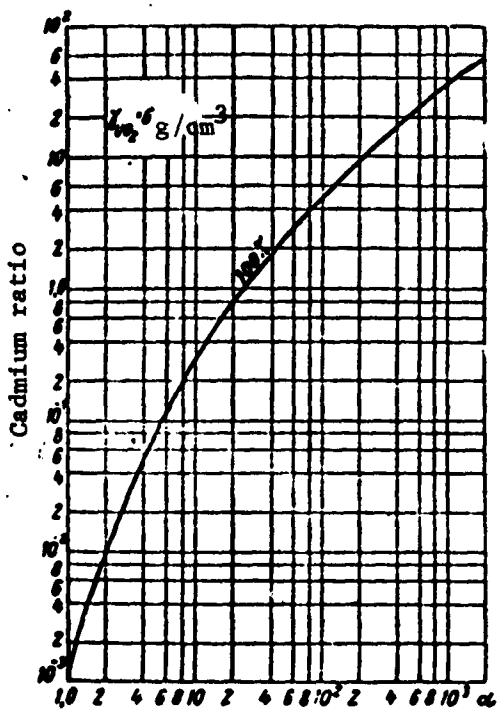

Fig. 7. Cadmium ratio as function of $\alpha = \rho_H/\rho_{U^{235}}$.

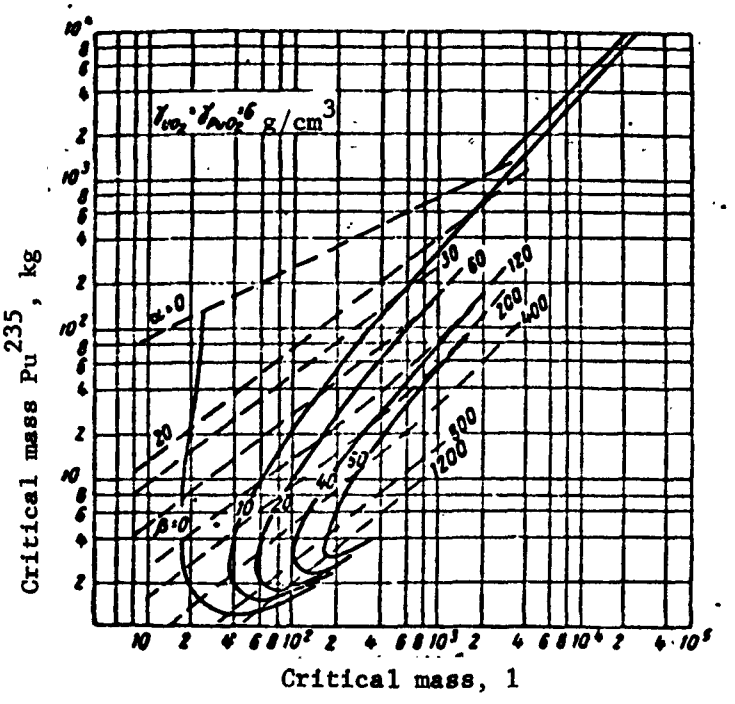

Fig. 8. Critical masses of spherical reactors UO_2 – PuO_2 – H_2O without reflectors at different ratios $\beta = \rho_U/\rho_{Pu}$.

The dashed lines join points with the same ratio $\alpha = \rho_H/\rho_{U^{235}}$.

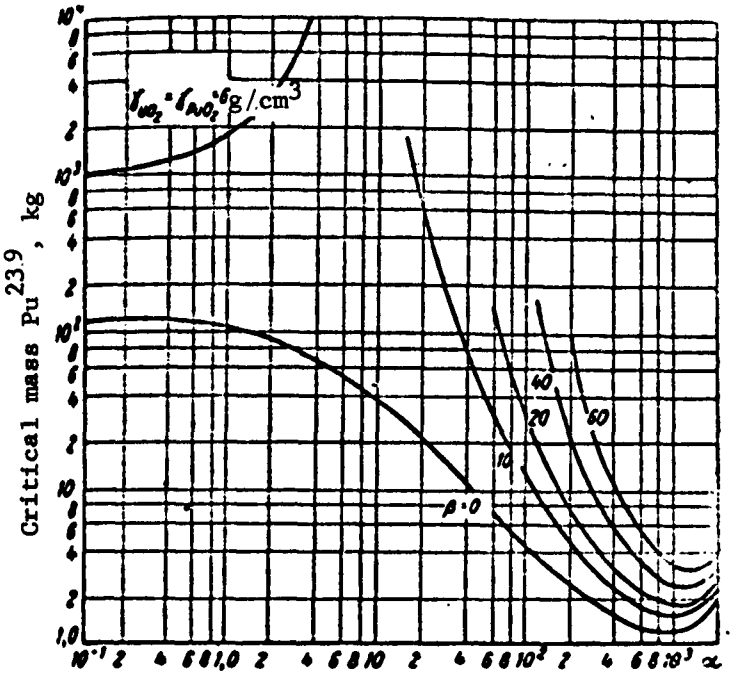

Fig. 9. Critical masses of spherical reactors UO_2 - PuO_2 - H_2O without reflectors at different ratios $\beta = \rho_U / \rho_{Pu}$ as a function of $\alpha = \rho_H / \rho_{Pu}$

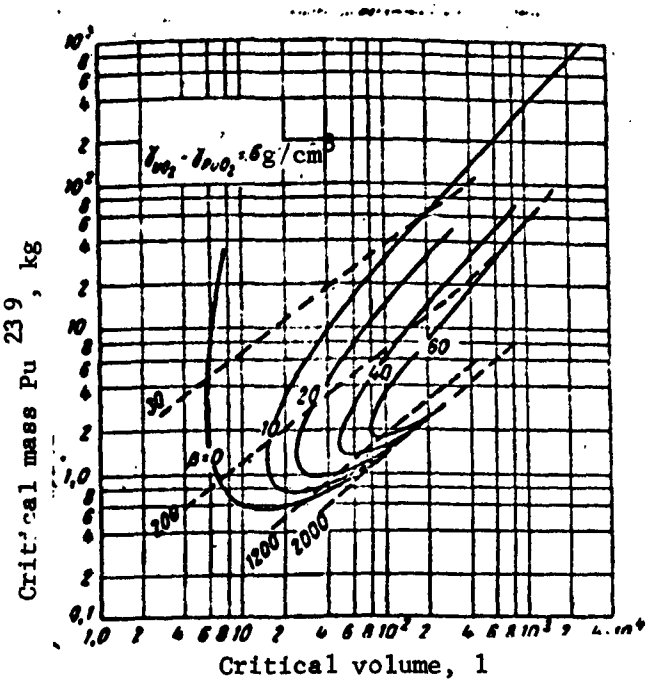

Fig. 10. Critical masses of spherical reactors UO_2-PuO_2-H_2O with a water reflector

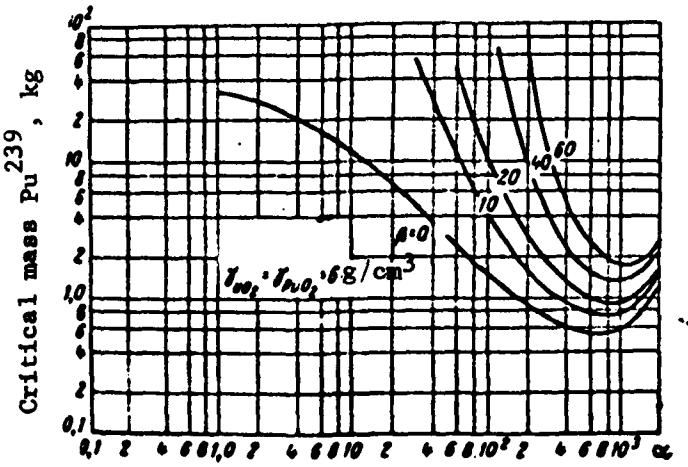

Fig. 11. Critical masses of spherical reactors UO_2 - PuO_2 - H_2O with water reflector as function от $\alpha = \rho_H/\rho_{Pu}$

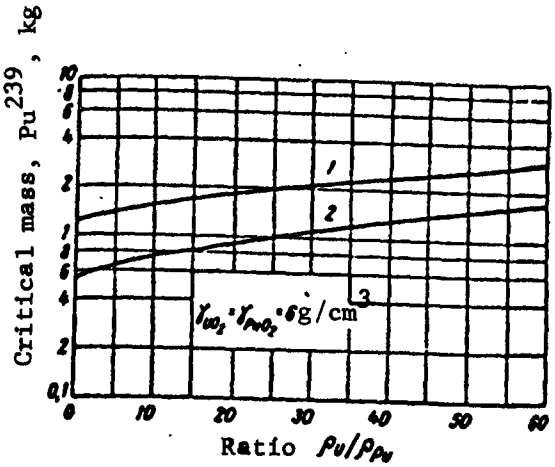

Fig. 12. Minimal critical masses of spherical reactors

$UO_2 - PuO_2 - H_2O$ as a function of $\beta = {}_{Pu}{}^{:}{}_{U}{}_{PH}$

1) without relector; 2) with water reflector.

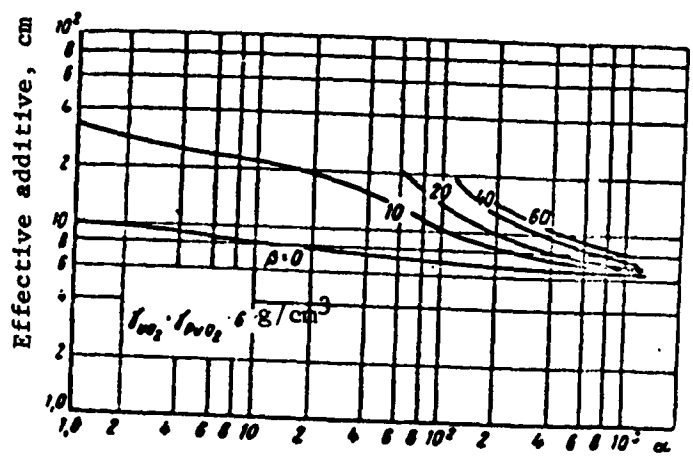

Fig. 13. Economy of water reflector for spherical reactors

$UO_2 - PuO_2 - H_2O$ as function of $\alpha = \rho / \rho_{Pu}$

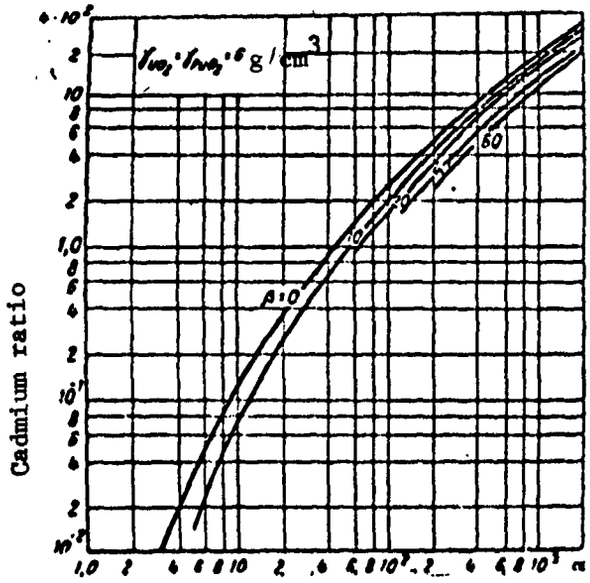

Fig. 14. Cadmium ratio as function of $\alpha = \rho_H/\rho_{Pu}$ at different $\beta = \rho_U/\rho_{Pu}$.

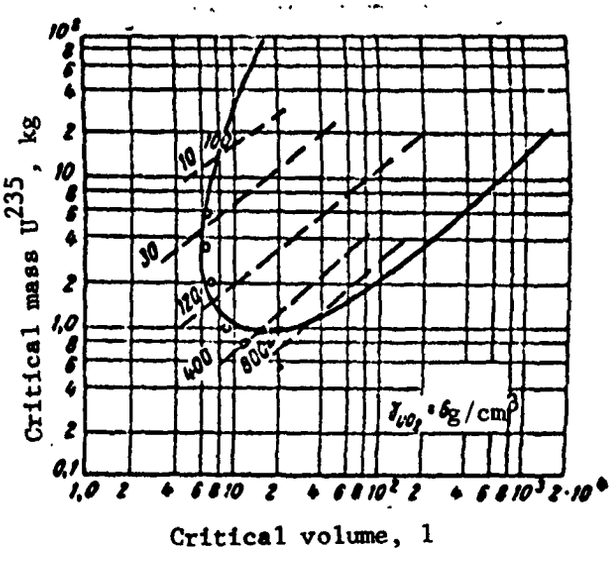

Fig. 15. Critical masses of spherical reactors $UO_2 - H_2O$ with a water reflector. Solid line shows theoretical data at 100% enrichment of uranium: o shows experimental data at 90% enrichment with uranium.

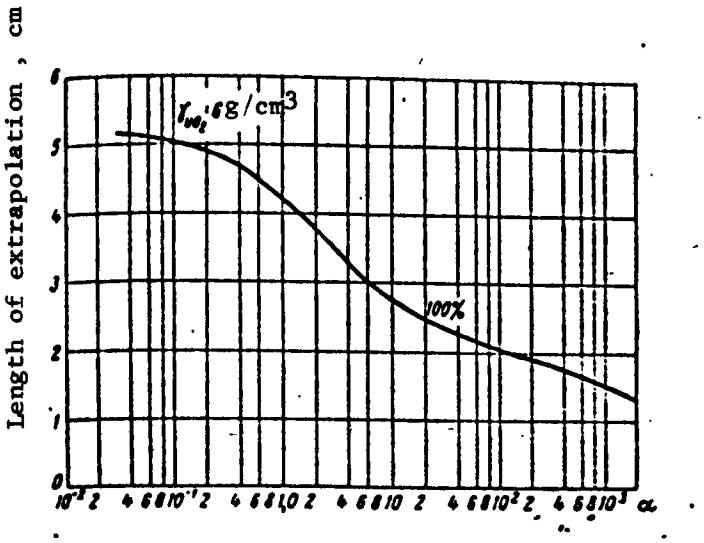

Fig. 16. Length of extrapolation $\underline{d} \simeq 0.71 \lambda_{tr}$ for spherical reactors UO_2 - H_2O at $\alpha = \rho_H/\rho_{U^{235}}$.

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INTERACTION OF SYSTEMS OF FISSILE MATTER IN A SCATTERING MEDIUM

by

V. G. Zagrafov

1. Approximate Solution of Integral Equation

The critical parameters of the system are determined by the Peierls integral equation /1/

$$\varphi(\vec{r}) = \int \frac{a(\vec{r}')}{\gamma(\vec{r}')} \varphi(\vec{r}') K(\vec{r}, \vec{r}')\, dV', \tag{1}$$

in which $\varphi(\vec{r})$ is the neutron distribution function;

$\lambda(\vec{r}) = N(\sigma_s + \sigma_f + \sigma_c)$ is the reverse free path length of the neutrons (N is the number of nuclei of matter in a unit of volume, and $\sigma_s, \sigma_f, \sigma_c$ are the scattering, fission and absorption cross sections);

$\gamma(\vec{r}) = \frac{\sigma_s + \sigma_f + \sigma_c}{\nu\sigma_f + \sigma_s}$ is the reciprocal mean neutron multiplication factor per one collision with a nucleus (ν is the number of neutrons occurring during one fission).

For fissile matter $\gamma < 1$, for absorptive matter $\gamma > 1$; during scattering without absorption $\gamma = 1$.

The kernel of Eq. (1) is determined by the expression

$$K(\vec{r}, \vec{r}') = \frac{\exp(-\int \varkappa dl)}{4\pi l^2}, \quad l = |\vec{r}' - \vec{r}|.$$

Integration is carried out with respect to the total volume of the system. For a system consisting of fissile matter (medium 1) and an absorptive matter (medium 2), Eq. (1) takes the following form

$$\varphi(\vec{r}) = \frac{1}{\gamma_1}\int_{V_1} \alpha(\vec{r}')\varphi(\vec{r}')K(\vec{r}, \vec{r}')dV' + \\ + \frac{1}{\gamma_2}\int_{V_2} \alpha(\vec{r}')\varphi(\vec{r}')K(\vec{r}, \vec{r}')dV',$$

in which $\underline{V}_1$ and $\underline{V}_2$ are the volumes of the regions filled with fissile and absorptive matter.

To obtain an approximate solution to Eq. (2), let us single out a spherical area of volume $\underline{V}$ in the center of the system which includes one of the fissile media and the area of the absorptive medium surrounding it. Let us term the singled out region the primary region, and let us call the remaining part of the volume the supplementing

region. Let us integrate Eq. (2) with respect to the volume of the primary region and assume $\varphi(\vec{r}) = \text{const}$*. We get

$$\frac{1}{V}\int_V dV \int_{V_1} \alpha(\vec{r}') K(\vec{r}, \vec{r}')\, dV' + \frac{c}{V}\int_V dV \int_{V_2} \alpha(\vec{r}') K(\vec{r}, \vec{r}')\, dV', \tag{3}$$

in which $c = \gamma_1/\gamma_2$.

Let us assume $dV' = l^2\, dl\, d\Omega$, in which $d\Omega$ is the solid angle at which the volume element dV' can be seen at point $\vec{r}'$ from point $\vec{r}$. By changing the order of integration, we can transform Eq. (3)

$$\gamma_1 = \frac{1}{V}\int \frac{d\Omega}{4\pi} \int_V dV \int_0^L (e^{-u_1} + ce^{-u_2})\, du + \\ + \frac{1}{V}\int \frac{d\Omega}{4\pi} \int_V dV \int_L^{u_-} (e^{-u_1} + ce^{-u_2})\, du, \tag{4}$$

* As shown in /2/, the error arising here has a sign favorable from the point of view of safety, and the absolute value of this error decreases as the volume of the primary region V increases.

~~** We will use the word optic for the distances expressed in~~

where $\underline{L}$ is the optic distance with respect to the ray $\vec{\Omega}$ from the point $\vec{r}$ lying in the primary region and the surface bounded by the primary region ;

$\underline{u} = \int \underline{a}dl$ is the optic distance with respect to the ray between the point $\vec{r}$ and the current point $\vec{r}'$;

$\underline{u}_\infty$ is the optic distance with respect to the ray $\vec{\Omega}$ between the point $\vec{r}$ and infinity;

$\underline{u}_1$ and $\underline{u}_2$ are the optic distances with respect to $\vec{\Omega}$ between the point $\vec{r}$ and a point $\vec{r}'$ lying in medium 1 or 2, respectively, so that the integrand is

$$e^{-u_1} + ce^{-u_2} = \begin{cases} e^{-u_1}, & \text{when } \vec{r}' \text{ in med. } 1, \\ ce^{-u_2}, & \text{when } \vec{r}' \text{ in med. } 2. \end{cases}$$

Let us designate $\underline{t} = \underline{u} - \underline{L}$; $\underline{t}' = \underline{u} - \underline{L}$; $\underline{t}'' = \underline{u} - \underline{L}$; $\underline{t}_\infty = \underline{u}_\infty - 1$ ($\underline{t}, \underline{t}', \underline{t}''$ and $\underline{t}_\infty$ are the optic distances with respect to the ray counted from the surface of the primary region).

Then

$$\int_L^{u_\infty} (e^{-u_1} + ce^{-u_2})\, du = e^{-L} \int_0^{t_\infty} (e^{-t'} + ce^{-t''})\, dt.$$

We will use the word 'optic' for the distances expressed in lengths of neutron free path.

Let us replace $\underline{t}'$ and $\underline{t}''$ in the integrand by the mean values of $\underline{t}'$ and $\underline{t}''$ over the volume of the primary region (the dependence of these values on the direction of $\vec{\Omega}$ is retained here). Let us consider a case inw which the primary region is symmetric (a sphere surrounded by a layer of absorptive). In this case, the values of the integrals

$$\int_V e^{-L} dV, \qquad \int_V dV \int_0^L (e^{-u_1} + ce^{-u_2})\, du$$

do not depend on the direction of $\vec{\Omega}$. Eq. (4) assumes the form

$$\gamma_1 = \frac{1}{V}\int_V dV \int_0^L (e^{-u_1} + ce^{-u_2})\, du + \frac{1}{V}\int_V e^{-L} dV \times$$
$$\times \int \frac{d\Omega}{4\pi} \int_0^{t_m} (e^{-\bar{t}'} + ce^{-\bar{t}''})\, dt.$$

(5)

If there is no supplementing region, the critical value γ_1^0 for the primary region, if $c = \gamma_1/\gamma_2$ is fixed, as determined by Eq. (5), is

$$\gamma_1^0 = \frac{1}{V}\int_V dV \int_0^L (e^{-u_1} + ce^{-u_2})\, du.$$

If there is no supplementing region, but the matter in the primary region is homogeneous in composition ($c=1$), the critical value γ^{00} found from Eq. (5) is

$$\gamma^{00} = \frac{1}{V}\int_V dV \int_0^L e^{-u}\, du = 1 - \frac{1}{V}\int_V e^{-L}\, dV.$$

Taking the values found for γ_1^0 and γ^{00} into account, the relationship for the critical parameter γ_1, given the fixed value $c = \gamma_1/\gamma_2$, assumes the following form

$$\frac{\gamma_1 - \gamma_1^0}{1-\gamma^{00}} = \int \frac{d\Omega}{4\pi} \int_0^{t_-} (e^{-\tau} + ce^{-\tau})\, dt. \qquad (6)$$

From now on we will take γ^0 and γ^{00} to mean the Eigenvalues for the isolated primary region, calculated by exact methods, for example, the improved diffusion method put forward by Romanov in Ref. /3/. (See p. 12 in this collection)

Finally, to simplify Eq.(6) let us use the fact that when averaging $\underline{t}'$ and $\underline{t}''$ over the volume $\underline{V}$ comes in the middle part of the cross section of the primary region the greatest weight /2/. Hence we will take it approximately that $\underline{t}'$ and $\underline{t}''$ in (6) coincide with $t_0'(\vec{\Omega})$ and $t_0''(\vec{\Omega})$ for a ray emerging from the center of the primary region.

Limiting Cases

Let us consider the limiting cases which are satisfied by Eq. 6:

1. Medium 2 is absent, $\gamma_1^0 = \gamma^{00}$

$$\int_0^{t_-} e^{-t_0'(\vec{\Omega})} dt = 1 - e^{-t_-(\vec{\Omega})}, \qquad \int_0^{t_-} e^{-t_0''(\vec{\Omega})} dt = 0.$$

The relationship (6) becomes a critical condition for the

homogeneous composition system obtained in /2/

$$\frac{1-\gamma_1}{1-\gamma_1^0}=\int e^{-t_\infty(\vec{\Omega})}\frac{d\Omega}{4\pi}. \qquad (7)$$

2. Media 1 and 2 are homogeneous in condition, $\underline{c}=1$. Eq. 6 gives us the critical relationship (7) for a homogeneous composition system.

3. Homogeneous system; the dimensions of bodies from fissile matter tend to 0 when their number is increased to an unlimited extent with retention of the relative volumes of the first and second media ($\underline{v}_1$ and $\underline{v}_2$). The radius of the primary region tends to 0, hence $\gamma_1^0 \to 0$ and $\gamma^\infty \to 0$. . The qualitative dependence of the integrand in Eq. 6 on the optic length $\underline{t}$ along the ray $\vec{\Omega}$ is shown in Fig. 1.

At the limit, when the increase in the number of bodies and the decrease in their dimensions and the distances between them are all infinite (i.e., when the relative volumes $\underline{v}_1$ and $\underline{v}_2$ are retained), the number of discontinuities in the integrand function at the intersection point of the ray and the interfaces

of the media per unit of length will increase to an unlimited extent. The relative dimensions of the areas of continuity, expressed in free path lengths, tend to $K\frac{\alpha_1 v_1}{\alpha_1 v_1 + \alpha_2 v_2}$ and $K\frac{\alpha_2 v_2}{\alpha_1 v_1 + \alpha_2 v_2}$ for segments of the ray lying in media 1 and 2, respectively; hence the integral of function $e^{-t_0'(\vec{\Omega})} + ce^{-t_0''(\vec{\Omega})}$ at the limit becomes the integral of the continuous function

$$\frac{\alpha_1 v_1 + c\alpha_2 v_2}{\alpha_1 v_1 + \alpha_2 v_2} e^{-t},$$

in which α_1 and α_2 are the reciprocal neutron free path lengths in media 1 and 2.

Eq. 6 takes the form

$$1 - \bar{\gamma} = \int e^{-t_\infty} \frac{d\Omega}{4\pi},$$

in which

$$\frac{1}{\bar{\gamma}} = \frac{1}{\gamma_1} \frac{\alpha_1 v_1}{\alpha_1 v_1 + \alpha_2 v_2} + \frac{1}{\gamma_2} \frac{\alpha_2 v_2}{\alpha_1 v_1 + \alpha_2 v_2},$$

(8)

coincides with the critical relationship (7) for a homogeneous system when the volume of the primary region tends to 0. The value of $\bar{\gamma}$ obtained is equal to the reciprocal mean nuetron multiplication factor for a homogeneous mixture of two substances in the proportion $\underline{v}_1 : \underline{v}_2$. Indeed, for a homogeneous mixture

$$\bar{\alpha} = \alpha_1 v_1 + \alpha_2 v_2; \quad \bar{\beta} = \beta_1 v_1 + \beta_2 v_2 = \frac{\alpha_1 v_1}{\gamma_1} + \frac{\alpha_2 v_2}{\gamma_2},$$

in which $\bar{\gamma} = \bar{\alpha}/\bar{\beta} = N(\nu\sigma_f + \sigma_s)$. It is easy to see that the mean value $\gamma = \alpha/\beta$ coincides with the value $\bar{\gamma}$ in Eq. (8).

4. No supplementing region. Eq. (6) gives us $\gamma_1 = \gamma_1^0$.

5. Medium 2 as an ideally scattering substance, $\underline{a}_2 = \infty$, $\gamma = 1$. We get $\int_0^{t_m} e^{-\alpha t}\, dt = 0$; $\int_0^{t_m} e^{-\alpha t}\, dt = 1$; the expression on the right-hand side of 6 is finite. Since for a medium in an ideally reflecting shell $\gamma_1^0 = 1$ and $\gamma^{00} = 1$, the expression in the left-hand side of 6 is finite at $\gamma_1 = 1$.

6. Medium 2 as an ideally absorbing substance, $\alpha_2 = \infty$. The value γ_1^0 determined by the exact method coincides in this case with the critical value γ for an isolated body without a shell. The expression on the right-hand side of the equation vanishes (since $\underline{c} \neq 0$), hence $\gamma_1 = \gamma_1^0$.

7. The distance between regions of fissile matter is infinitely increased, the density of medium 2 remaining the same. Here $\gamma^{\infty} \to 1$, since the radius of the primary sphere tends to infinity. In order to to keep Eq. (6) the left-hand side finite, γ_1 should tend to γ_1^0. When determined by exact methods, γ_1^0 coincides with the critical value γ for a body in an infinite medium with the constant $\gamma_2 = \frac{\gamma_1}{e}$.

Convolution of Eq. (6) for System of Spheres

Let us apply Eq. 6 to a system of spheres with radius R located at equal intervals around the volume of a sphere with radius $\underline{a}^{\infty}$ filled with an absorptive (or scattering) material. Let us single out the primary sphere containing the central sphere in the center of the system.

Let us divide up the integration interval with respect to the beam $(0; \underline{t}_{\infty})$ into segments, in each of which the integrand function in (6) is monotonic. The boundary points of these segments are the points at which the ray $\vec{\Omega}$ intersects with the interfaces of media 1 and 2. Let us designate the optical distances with respect to the ray between the surface of the primary sphere and these points as $\underline{t}_1$, $\underline{t}'_1$, $\underline{t}_2$, $\underline{t}'_2$ ..., $\underline{t}_k$, $\underline{t}'_k = \underline{t}_{\infty}$, in which $\underline{k}$ is the number

of spheres intersected by the ray (see Fig. 1). Then

$$\int_0^{t_\infty}(e^{-t_0'}+ce^{-t_0''})\,dt = c+(1-c)e^{-t_1}-(1-c)e^{-t_1'}+$$
$$+(1-c)e^{-t_2}-\cdots+(1-c)e^{-t_k}-e^{-t_k'}.$$

(9)

Let us introduce the mean values:

$\tau_{\hat{m}}$ is the mean optic distance of the path in the sphere,

and

$\eta_{\hat{m}}$ is the mean optic distance in medium 2 with respect to the ray $\vec{\Omega}$ in the surfaces of the two neighboring spheres.

When the geometric progression in Eq. 9 has been folded and the mean lengths have been introduced, Eq. (6) takes the form

$$\frac{\gamma_1-\gamma_1^0}{1-\gamma^0}=(1-e^{-t_\infty})\left[1-(1-c)\frac{(1-e^{\eta_{cp}})}{(1-e^{-(\tau+\eta)_{cp}})}\right].$$

(10)

Let us supplment Eq. 10 with relationships for t, $\tau_{\hat{m}}$ and $\eta_{\hat{m}}$ for a case in which the scattering matter is uniformly distributed in the space between the spheres. Let us represent the

optical radius of the system as $\underline{t}_{\infty} = \overline{\alpha}\underline{a}$, in which $\overline{\alpha}$ is the mean probability of extracting neutrons from the beam per unit of path length. If there is no medium 2, then

$$\bar{a}_1 = pS[1 - g(a_1 R)],$$

in which $\underline{p}$ is the number of spheres per unit of volume,

$\underline{S}$ is the cross section of the sphere, and

$\underline{g}(\alpha_1 \underline{R})$ is the transmittance coefficient of the sphere for a plane neutron beam determined by the following relationship /2/

$$g(a_1 R) = \bar{e}^{\tau_{cp}} \frac{1}{S} \int \exp(-\tau)\, dS = \frac{1 + (1+2a_1 R)\exp(-2a_1 R)}{2(a_1 R)^2}. \quad (11) \qquad (11)$$

If there are no spheres, the probability of extracting neutrons per unit length may be represented as the difference

$$\bar{a}_2 = a_2 - pS[1 - g(a_2 R)].$$

where $\underset{2}{\alpha}$ is the probability of extracting neutrons in medium 2;

$pS[1-g(a_2 R)]$ is the probability of extracting neutrons in the sphere system from medium 2, identical in geometric size and position

to the spheres of fissile matter.

The total probability of extracting neutrons from the beam per unit length is equal to

$$\bar{a} = \bar{a}_1 + \bar{a}_2 - a_2 + pS[g(a_2 R) - g(a_1 R)]. \tag{12}$$

The values $\eta_{\hat{m}}$ and $(\eta+\tau)$ in Eq. (10) are determined by the relationships

$$e^{-\eta_{\hat{m}}} = \frac{1}{S}\int e^{-\eta} dS = e^{-\Lambda}\frac{1}{S}\int e^{(\Lambda-\eta)} dS = e^{-\Lambda} g(-a_2 R),$$
$$e^{-(\tau+\eta)_{\hat{m}}} = \frac{1}{S} e^{-\Lambda}\int e^{(\Lambda-\eta-\tau)} dS = e^{-\Lambda} g[(a_1 - a_2) R],$$

in which $\Lambda = a_2/pS$ is the path length expressed in the neutron free path length in medium 2 per one intersection of the sphere by the ray $\vec{\Omega}$; the function $\underset{\sim}{g}(\underset{\sim}{x})$ is determined by Eq. (11). We should point out that the function $g(x)$ satisfies the following approximate relationships

$$g(-x) \approx [g(x)]^{-1}, \quad g(x_1 - x_2) \approx \frac{g(x_1)}{g(x_2)}.$$

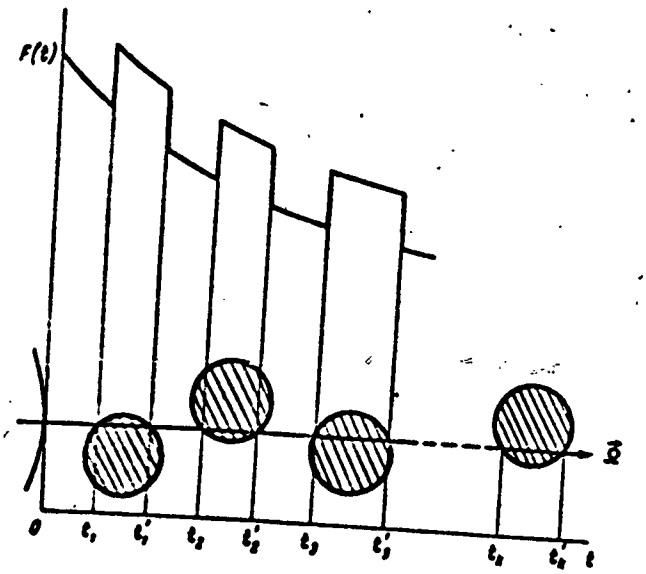

Fig. 1. Qualitative dependence of function $F = e^{-t_0'} + ce^{-t_1'}$ on the optical distance t in the direction of the ray $\vec{\Omega}$.
Medium I (circles) is dashed.

which is equivalent to

$$e^{-(\eta+\tau)_{cp}} \approx e^{-\eta_{cp}} e^{-\tau_{cp}}.$$

Finding the Critical Parameters γ_1^0 and γ^{00} By the Asymptotic Diffusion Method

Let us find the critical relationship for a sphere from medium 1 with radius $\underline{R}$ surrounded by a shell of matter 2 with an external radius $\underline{r}_0$ (isolated primary region). The asymptotic solution of the diffusion equation satisfying the condition of finiteness of the neutron density is $\varphi_{ac}(r)$ at $\underline{r} = 0$, and the condition at which $\varphi_{ac}(r)$ vanishes on the extrapolated external boundary of the shell $\underline{r}_e = \underline{r}_0 + 0.71\,\gamma_2/\alpha_2$ takes the form

$$\varphi_{ac1}(r) = \frac{1}{r}\sin \alpha_1 k(\gamma_1) r;$$

$$\varphi_{ac2}(r) = \begin{cases} \frac{1}{r}\operatorname{sh} \alpha_2 k(\gamma_2)(r_e - r) & \text{at } \gamma_2 > 1 \\ \frac{1}{r}\sin \alpha_2 k(\gamma_2)(r_e - r) & \text{at } \gamma_2 < 1. \end{cases}$$

in which the dependence $\underline{k}(\gamma)$ is determined by the relationships $\underline{k} = \tan(\underline{k}\gamma)$ at $\gamma < 1$ and $\underline{k} = \text{th}(\underline{k}\gamma)$ at $\gamma > 1$ //4/ (also see Romanov's paper in this collection).

Let us assume that at the interface of the two media the asymptotic solution φ_{ac1} and φ_{ac2} satisfy the boundary condition which takes into account the deviation φ_{ac} near the boundary from the true density $\varphi(\underline{r})$, satisfying Eq. (1). In its simplest form the approximate boundary condition derived by Davison /3/ takes the form

$$\frac{\text{grad}\,\varphi_{ac1}}{a_1\varphi_{ac1}} = \frac{\text{grad}\,\varphi_{ac2}}{a_2\varphi_{ac2}} \quad \text{at} \quad r = R.$$

Substituting φ_{ac} into the boundary relationship, we obtain the critical condition

$$\frac{1}{a_1}[1 - a_1 k(\gamma_1) R \operatorname{ctg} a_1 k(\gamma_1) R] = \begin{cases} \frac{1}{a_2}[1 + a_2 k(\gamma_2) R \operatorname{cth} a_2 k(\gamma_2)(r_0 - R)] & \text{at } \gamma_2 > 1, \\ \frac{1}{a_2}[1 + a_2 k(\gamma_2) R \operatorname{ctg} a_2 k(\gamma_2)(r_0 - R)] & \text{at } \gamma_2 < 1. \end{cases} \quad (13)$$

We can obtain a relationship for the critical parameter of the sphere γ_1^0 at a fixed ration $c = \gamma_1/\gamma_2$ by replacing γ_1 by γ_1^0 and γ_2 by γ_1^0/c in Eq.(13). If we assume $\gamma_1=\gamma_2=\gamma^{00}$, , Eq.(13) gives us a relationship determining the critical parameter for a homogeneous composition of the matter in the primary sphere.

In practice the asymptotic diffusion method is used for approximate calculations when the radius of curvature of the interface is either large, or comparable with the neutron free path length in the media (roughly at $R > 0.3\alpha^{-1}$). If this condition is not satisfied, the diffusion method may prove unreliable.

Example 1. Spheres of Ou(93.5)* with radius R = 4 cm are uniformly spread through an infinite space in a scattering medium with constant absorption γ_2 = 1.1 and free path lengths α_2^{-1} = 10 cm. We are to find the critical distance between the spheres. The parameters for Ou (93.5): are: γ_1 = 0.74, α_1 = 0.25 cm^{-1} **.

* Ou (93.5) is metallic uranium containing 93.5% isotope U^{235}.

** The values of these parameters are found in accordance with experimental data by measuring the critical masses given in /5/.

Let us single out the primary spherical region of the greatest possible radius in such a way that it includes one of the spheres. If there are p spheres per unit of volume, the radius of the primary sphere $r_0 = d - R$, in which $d = p^{-\frac{1}{3}}$ is the distance between the centers of the spheres. The condition of the problem is such that we should take $t_\infty = \infty$ in Eq. (6). The system of transcendental equations (6) and (13) can be solved by the selection method r_0. For example, at $r_0 = 22.6$ cm, Eq. (13) gives us $\gamma_1^0 = 0.553$, $\gamma^{00} = 0.756$, and Eq. (6) is satisfied if we assume that $c = 0.769$, instead of the true value $c = \gamma_1/\gamma_2 = 0.673$. At $r_0 = 16.6$ cm, $\gamma_1^0 = 0.535$ and $\gamma^{00} = 0.683$, Eq. (6) is satisfied at $c = 0.647$. At $r_0 = 18$ cm, $\gamma_1^0 = 0.538$ and $\gamma^{00} = 0.697$, Eq. (6) is satisfied at $c = 0.669$, and so on.

By interpolating we find that the true value corresponds to $r_0 = 17.8$ cm, i.e., the critical distance between the spheres $d = r_0 + R = 21.8$ cm. We should point out that in our example the ratio $(1 - e^{-\eta/m})/(1 - e^{-(\eta+\tau)/m})$ virtually coincides with unity in Eq. 10. The difference between this ratio and unity is a small correction

Comparative table of physical parameters of system

of spheres and homogeneous fissile medium

Fissile material medium	System of spheres of fissile material
Probability of fission per unit path length $a_f = N\sigma_f$	Probability of interaction with spheres per unit $\tilde{p}$th length $pS[1-g(a_1R)]$
Reciprocal free path length $a = N(\sigma_f + \sigma_s + \sigma_c)$	Probability of extraction of neutrons from beam per unit path length $\bar{\alpha}$ /given uniform distribution of scattering, $\bar{\alpha}$ is depermined by (12)/
Probability of scattering per unit path length $a_s = N\sigma_s$	$\frac{1}{\gamma_2}[\bar{a} - pS(1-g)]$
Probability of absorption per unit path length $a_c = N\sigma_c$	$\left(1-\frac{1}{\gamma_2}\right)[\bar{a} - pS(1-g)]$
Breeding coefficient per one fission ν	Coefficient of multiplication by sphere of neutron flux incident on sphere and undergoing at least one collision with nuclei of matter in sphere, Q
$\beta = N(\nu\sigma_f + \sigma_s)$	$\left(Q-\frac{1}{\gamma_2}\right)pS(1-g)+\frac{1}{\gamma_2}\bar{a}$
Breeding coefficient for neutrons per one collision $\frac{1}{\gamma} = \frac{\nu\sigma_f + \sigma_s}{\sigma_s + \sigma_f + \sigma_c}$	$\left(Q-\frac{1}{\gamma_2}\right)\frac{pS(1-g)}{\bar{a}}+\frac{1}{\gamma_2}$

in many of the problems met with in practice. This fact relieves of the need to calculate η_m and $(\eta+\tau)_{cp}$, exactly, when the scattering matter is nonuniformly distributed in the space between the spheres, and when an exact determination of these values may prove laborious.

II. Kinetic Equation for Averaged Neutron Density

The breeding of neutrons in a system consisting of a large number of spheres multiplying a stream of neutrons inmpinging upon them is similar to the microscopic process of neutron breeding in a fissile medium. A parallel can be drawn between the concept and the physical values in these cases (see Table).

On account of the analogy, the Boltzmann kinetic equation can be used approximately to describe the neutron transfer in the sphere system, if we replace the elementary constants by effective microscopic constants, in accordance with the Table. Here we have to take into account two spherical anomalies in our problem. The first is the nonisotropic nature of the angular distribution of a neutron flux yielded from the sphere when the incident, plane neutron flux breads. A similar phenomenon during breeding of neutrons in a fissile matter would be the nonisotropic nature of the angular distribution of neutrons yielded during fission. Hence in order to

use the transfer equations for the problem of a system of bodies we have to generalize the concept of transport path for the case of nonisotropic neutron yield during fission.

If $N(\vec{r},\vec{\Omega})\,d\Omega$ is the number of neutrons per unit of volume at the point $\underline{r}$, whose directions of motion lie in the element of the solid angle $d\Omega$ around $\vec{\Omega}$, the kinetic equation for $N(\vec{r},\vec{\Omega})$ takes the following form

$$v\vec{\Omega}\,\mathrm{grad}\,N(\vec{r},\vec{\Omega}) + v\alpha N(\vec{r},\vec{\Omega}) = v\int N(\vec{r},\vec{\Omega}')\beta f(\vec{\Omega}'\to\vec{\Omega})\,d\vec{\Omega}' + \frac{q(r)}{4\pi}, \tag{14}$$

in which $\beta f(\vec{\Omega}'\to\vec{\Omega})\,d\Omega'$ is the probable number of secondary neutrons with a direction of motion $\vec{\Omega}$, appearing during interaction between the matter and the neutron with a direction of motion in the elementary cone $d\Omega$ around $\vec{\Omega}$, per unit of its path length;

$q(\vec{r})$ is the number of neutrons from the source per unit of volume and unit time.

If the angular distribution of the neutrons after scattering and fission is nonisotropic, then

$$\beta f(\vec{\Omega}' \to \vec{\Omega}) = \nu_2 f_f(\vec{\Omega}' \to \vec{\Omega}) + \alpha_s f_s(\vec{\Omega}' \to \vec{\Omega}),$$

in which $f_f(\vec{\Omega}' \to \vec{\Omega})$ and $f_s(\vec{\Omega}' \to \vec{\Omega})$ are angular distribution functions of the probable number of neutrons after fission and scattering, satisfying the normalization condition.

$$\int f_f(\vec{\Omega}' \to \vec{\Omega})\, d\Omega = 1; \quad \int f_s(\vec{\Omega}' \to \vec{\Omega})\, d\Omega = 1.$$

By integrating Eq. 14 with respect to $d\Omega$, we obtain the continuity equation

$$\operatorname{div} \vec{j}(\vec{r}) = (\beta - \alpha)\, \upsilon n(\vec{r}) + q(\vec{r}), \tag{15}$$

in which $n(r) = \int N(\vec{r}, \vec{\Omega})\, d\vec{\Omega}$ — is the neutron density, and

$\vec{j}(\vec{r}) = \upsilon \int \vec{\Omega} N(\vec{r}, \vec{\Omega})\, d\Omega$ — is the neutron flux density.

Let us multiply Eq. (14) by Ω and integrate with respect to $d\Omega$. Taking it into account that the functions f_f and f_s only depend on the angle between $\vec{\Omega}$ and $\vec{\Omega}'$, we get

$$\upsilon \int \vec{\Omega} [\vec{\Omega} \operatorname{grad} N(\vec{r}, \vec{\Omega})]\, d\Omega + \alpha_{tr} \vec{j} = 0, \tag{16}$$

in which

$$\alpha_{ir} = \alpha_s(1-\bar{\mu}_s) + \alpha_f(1-\nu\bar{\mu}_f) + \alpha_c,$$
$$\bar{\mu}_s = \int (\vec{\Omega}\vec{\Omega}') f_s(\vec{\Omega}\vec{\Omega}')\, d\Omega', \quad \bar{\mu}_f = \int (\vec{\Omega}\vec{\Omega}') f_f(\vec{\Omega}\vec{\Omega}')\, d\Omega';$$

$\bar{\mu}_s$ and $\bar{\mu}_f$ are the mean cosines of angular distribution of the neutrons after scattering and fission.

The continuity Eqs. (15) and (16) coincide with the relevant equations for isotropic neutron distribution during scattering and fission, provided the latter undergo substitution of α by $\alpha_{ir} = \alpha_s(1-\bar{\mu}_s) + \alpha_f(1-\nu\bar{\mu}_f) + \alpha_c$, and β by $\beta_{ir} = \nu\alpha_f(1-\bar{\mu}_f) + \alpha_s(1-\mu_s)$. In the problem of a system of interacting spheres, it is essential to take the nonisotropic neutron yield from the spheres into account, since this effect is unfavorable from the viewpoint of the safe handling of fissile matter (later on we will show that the mean cosine of angular distribution when the neutrons are bred by a sphere is always negative).

The second anomaly in our problem, also unfavorable from the viewpoint of safety, is the deviation of the true neutron density distribution in the scattering medium from the mean value $\vec{n}(\vec{r})$. If the free path length in the scattering medium is small compared with

the distance between the spheres, a neutron ~~emerging~~ yielded from the sphere has the probability of going back to the same sphere after scattering and breeding again, Thus, some of the neutrons coming from the sphere cause an infinite chain of successive multiplications. Since the scattering of those which have emerged and returned to the same sphere occurs mainly in the close-lying layers of the scattering medium, the phenomenon leads to an increase in the neutron concentration near these spheres, compared with the mean density in the spaces between them $\underline{n}(\vec{r})$, contained in the continuity ~~equation~~ Eq. (15).

When these two effects have been taken into account, all the approximate methods of solving the transfer equations (for example, the asymptotic diffusion method) can be applied to the problem of the system of interacting bodies . Consideration for these effects is made below. Let us first look at a case in which the effects are only slight.

Example 2. Let us evaluate the critical number of spheres

SEE PAGE 127 (16)

x This relationship is often called the generalized Fick law /3/.

of Ou(9.35) of mass m uniformly distributed in the empty cavity of a spherical reflector with reflection coefficient (Albedo) A ≲ 0.8 - 0.9. Volume of the cavity is V_0. Let us assume that there is no absorptive or scattering matter in the cavity, that the radius of the spheres R is small compared with the critical radius R_0 of an isolated sphere of Ou (9.35).

Applying the asymptotic diffusion method, we find the critical relationship. If we are given the Albedo of the reflecting shell, the boundary condition on the internal surface of the reflector takes the form [3]

$$-\frac{\operatorname{grad}\varphi_{ac}}{a_{tr}\varphi_{ac}}=\frac{3}{2}\frac{1-A}{1+A}.$$

Substituting the equation for φ_{as} in the active zone (sphere system), we obtain the critical relationship

$$\frac{1}{a_{tr}a}(1-a_{tr}ka\operatorname{ctg}a_{tr}ka)=\frac{3}{2}\frac{1-A}{1+A}, \qquad (17)$$

in which a is the radius of the system. For the case in point, in which the Albedo is close to unity, and the coefficient of neutron

breeding by each sphere is considerably less than unity, the following decompositions hold

$$\operatorname{ctg}\alpha_{1r}ka \approx \frac{1}{\alpha_{1r}ka} - \frac{\alpha_{1r}ka}{3}, \qquad k^2 \approx 3\left(\frac{\beta_{1r}}{\alpha_{1r}} - 1\right). \tag{18}$$

Taking into account that $\beta_{1r} - \alpha_{1r} = \beta - \alpha$, we can represent Eq. (17) with consideration for (18) in the following form

$$pS(1-g)(Q-1)a = \frac{3}{2}\frac{1-A}{1+A}.$$

Let us introduce the multiplication factor $\underline{Q}^*$, which is defined as the ratio of the flux of neutrons leaving the sphere to the flux of neutrons incident on it, irrespective of whether the neutrons striking the sphere have reacted with the matter or passed through it without undergoing any collisions. The values $\underline{Q}$ and $\underline{Q}^*$ are related as follows

$$Q^* - 1 = [1 - g(\alpha, R)](Q-1). \tag{20}$$

As will be shown later, $\underline{Q}^*$ can be approximately described by

the following interpolation between the extreme cases $\underline{R} \to 0$ and $\underline{R} \to \underline{R}_0$

$$\frac{1}{Q^*} = 1 - \frac{R}{R_0}. \tag{21}$$

Using Eqs. (20) and (21), and expressing all values in Eq. (19) in terms of the sphere mass $\underline{m}$, the volume of the system $\underline{V}_0$ and the total number of spheres in the system $\underline{N}_0$, we get the critical condition

$$N_0 = 2\left[\left(\frac{M}{m}\right)^{1/3} - 1\right]\left(\frac{\rho V_0}{m}\right)^{2/3}\frac{1-A}{1+A}, \tag{22}$$

in which $\underline{M}$ is the critical mass of the isolated sphere (for Ou (93.5) $\underline{M}$ = 51.3 kg /5/), ρ is the density of the matter in the spheres (for Ou (93.5) is equal to 18.8 g/cm^3).

We should point out that the critical Eqs. (19) and (20) do not include dependence on the angular distribution of the neutron flux from the spheres. The reason for this is that on account of the proximity of the albedo of the reflector $\underline{A}$ and unity, the

mean neutron density varies slightly along the radius of the system, and allowance for the anisotropy of the angular distribution of the flux during breeding gives a correction for the small value of the gradient of the mean neutron density (if the latter is constant, for example, in an infinitely-extended system, the anysotropy has no effect at all on the state of the system).

Multiplication of Instant Neutron Stream by Body Made of Fissile Matter

Let us derive relationships for the multiplication factor for a plane beam of neutrons Q and mean cosine $\bar{\mu}$ of angular distribution of the yield flux. All the elementary processes in the body (scattering of neutrons by nuclei and fission) are considered isotropic, hence in the kinetic Eq. (14) we must assume $f(\vec{\Omega}' \cdot \vec{\Omega}) = \frac{1}{4\pi}$ according to normalization of the function f. We will take the isotropic sources in Eq. (14) to mean secondary neutrons occurring through interaction of the incident plane beam* and the matter . Integrating Eq. (14) along the ray emerging from point r in the direction Ω we get

* The method used here of reducing the problem with anisotropic source to isotropic distribution of sources is described in /6/.

$$4\pi v N(\vec{r}, \vec{\Omega}, \vec{\Omega}_0) = \int_0^{\bar{l}} \beta(\vec{r} - l\vec{\Omega})\, n(\vec{\Omega}_0, \vec{r} - l\vec{\Omega}) \exp\left(-\int_0^{l} \alpha(\vec{r} - l'\vec{\Omega})\, dl'\right) dl + \int_0^{\bar{l}} q(\vec{\Omega}_0, \vec{r} - l\vec{\Omega}) \exp\left[-\int_0^{l} \alpha(\vec{r} - l'\vec{\Omega})\, dl'\right] dl, \quad (23)$$

(23)

in which $\vec{\Omega}_0$ is the direction of motion of the neutrons in the plane beam;

l is the coordinate along the ray $\vec{\Omega}$.

We can obtain an expression for the neutron flux $J(\vec{\Omega},\vec{\Omega}_0)$, emerging from the body in the direction $\vec{\Omega}$ by integrating Eq. (23) with respect to the area of the cross section normal to $\vec{\Omega}$

$$4\pi J(\vec{\Omega}, \vec{\Omega}_0) = 4\pi v \int N(\vec{r}, \vec{\Omega}, \vec{\Omega}_0)\, dS = \int_V \beta(\vec{r})\, v n(\vec{r}, \vec{\Omega}_0) \times \exp\left(-\left|\int_0^{l} \alpha\, dl'\right|\right) dV + \int_V q(\vec{r}, \vec{\Omega}_0) \exp\left(-\left|\int_0^{l} \alpha\, dl'\right|\right) dV,$$

in which $\left|\int_0^{l} \alpha\, dl'\right|$ is the optical distance between point r and the surface of the body along the ray $\vec{\Omega}$, and V is the volume

of the body.

Let us designate $\chi(\vec{\Omega}, \vec{r}) = \exp\left(-\int_0^L \alpha dl'\right)$. If the body is impinged upon by a single (per unit of area) plane neutron stream in the direction $\vec{\Omega}_0$, the density of the sources $q(\vec{r}, \vec{\Omega}_0)$, according to the definition given above, is equal to

$$q(\vec{r}, \vec{\Omega}_0) = \beta(\vec{r})\chi(-\vec{\Omega}_0, \vec{r}).$$

The expression for the flux takes the following form:

$$4\pi J(\vec{\Omega}, \vec{\Omega}_0) = \int \beta(\vec{r}) vn(\vec{r}, \vec{\Omega}_0)\chi(\vec{\Omega}, \vec{r})dV + \\ + \int \beta(\vec{r})\chi(\vec{\Omega}, \vec{r})\chi(-\vec{\Omega}_0, \vec{r})dV.$$

For a homogeneous body of spherical shape

$$4\pi J(\vec{\Omega}, \vec{\Omega}_0) = \iint \beta vn[\vec{r}, (\vec{\Omega}'\vec{\Omega}_0)]\chi[\vec{r}(\vec{\Omega}'\vec{\Omega})]r^2 dr d\Omega' + \\ + \iint \beta\chi[r, (\vec{\Omega}'\vec{\Omega})]\chi[r, -(\vec{\Omega}'\vec{\Omega}_0)]r^2 dr d\Omega', \quad (24)$$

in which $(\vec{\Omega}\vec{\Omega}_0)$ is the cosine of the angle between the unit vectors

$\vec{\Omega}$ and $\vec{\Omega}_0$.

The multiplication coefficient defines the number of neutrons leaving the body per one act of interaction between the matter and the neutrons striking the body is equal to

$$Q = \frac{\int J(\vec{\Omega}\,\vec{\Omega}_0)\,d\Omega}{\int \varepsilon(\vec{r})\,\chi(\vec{r}, -\vec{\Omega}_0)\,dV} = \frac{\int \beta\bar{\upsilon}\bar{n}(r)\,\bar{\chi}(r)\,r^2dr + \int \overline{\beta\chi}^2(r)\,r^2dr}{\int \alpha\bar{\chi}(r)\,r^2dr}, \tag{25}$$

in which

$$\bar{\chi}(r) = \int \chi[r, (\vec{\Omega}'\vec{\Omega})]\,\frac{d\Omega}{4\pi}, \quad \bar{n}(r) = \int n[r, (\vec{\Omega}'\vec{\Omega})]\,\frac{d\Omega}{4\pi}.$$

The mean cosine $\bar{\mu}$ of the neutron flux emerging from the body is

$$\bar{\mu} = \frac{\int (\vec{\Omega}\,\vec{\Omega}_0)\,J(\vec{\Omega}\vec{\Omega}_0)\,d\Omega}{\int J(\vec{\Omega}\,\vec{\Omega}_0)\,d\Omega} = \frac{\int \beta\upsilon\,\overline{\mu n}(r)\,\bar{\chi}(r)\,r^2dr - \int \beta[\bar{\chi}(r)]^2 r^2dr}{Q\int \alpha\bar{\chi}(r)\,r^2dr}, \tag{26}$$

in which

$$\bar{\chi}(r) = \int (\vec{\Omega}'\vec{\Omega})\,\chi[r\,(\vec{\Omega}'\vec{\Omega})]\,\frac{d\Omega}{4\pi}, \quad \overline{\mu n}(r) = \int (\vec{\Omega}'\vec{\Omega})\,n\,[r, (\vec{\Omega}'\vec{\Omega})]\,\frac{d\Omega}{4\pi}.$$

When deriving Eq. (26) we made use of the following relationship,

valid for the arbitary function $\Psi(\vec{\Omega}'\vec{\Omega})$:

$$\int \vec{\Omega}\Psi(\vec{\Omega}'\vec{\Omega})\,d\Omega = \vec{\Omega}' \int (\vec{\Omega}'\vec{\Omega})\,\Psi(\vec{\Omega}'\vec{\Omega})\,d\Omega.$$

In a limiting case, in the radius of the sphere R is small compared with the critical radius R_0 and with the free path length of the neutrons, we get $\nu n \ll \chi$, $\chi \approx 1$. Confining ourselves to the first terms in the expansion of equalities (25) and (26) into a power series R, we get

$$Q_1 = 1, \quad \bar{\mu} = -\frac{1}{15} R^2. \tag{27}$$

Let us consider another limiting case in which the radius of the sphere R is close to the physical radius R_0. Let us expand the neutron flux vn in Eq. (25) into a series with respect to the eigen functions φ_i of the Peierls homogeneous Eq. (1). Omitting the operations described in /6/, we get

$$\nu n[r, (\vec{\Omega}'\vec{\Omega}_0)] = \sum_i [\nu n]_i \varphi_i[r, (\vec{\Omega}'\vec{\Omega}_0)],$$

$$[\nu n]_i = \frac{1}{\lambda_i - 1} \int \beta \varphi_i[r, (\vec{\Omega}'\vec{\Omega}_0)]\,\chi[r, -(\vec{\Omega}'\vec{\Omega}_0)]\,dV.$$

in which λ_i are the Eigenvalues of Eq. (1) ($i = 0, 1, 2, \ldots$). The Eigen functions φ_i form an orthogonal and normal system

$$\int \beta \varphi_i \varphi_j dV = \begin{cases} 1 & \text{at } i=j, \\ 0 & \text{at } i \neq j. \end{cases}$$

Confining ourselves in the expansion to the two first terms containing the function φ_0, which is not dependent on $(\Omega'\Omega)$, and φ_1 which is proportional to the first power $(\vec{\Omega}'\vec{\Omega}_0)$, and taking it into account that $\bar{\varphi}_0(r) = \varphi_0(r)$, $\bar{\varphi}_1(r) = 0$, we obtain the following for a sphere close to the critical state

$$Q = \frac{1}{\lambda_0 - 1} \frac{\left(\int \beta \varphi_0(r) \bar{\chi}(r) r^2 dr\right)^2}{\int \chi(r) r^2 dr \int \beta \varphi_0^2(r) r^2 dr}.$$

(27')

Let us approximately replace the distribution of asymptotic density $\varphi_{0\,ac}(r) = \frac{\sin \alpha r}{\alpha r}$ by the parabolic dependence $\varphi_0(r) = 1 - \left(\frac{r}{R_{0e}}\right)^2$, in which R_{0e} is the extrapolated critical radius of the sphere determined

in the asymptotic diffusion theory by the relationships*

$$R_{0\gamma} = R_0 + 0{,}71\gamma = \frac{\pi}{k}, \quad k = \operatorname{tg}(k\gamma).$$

Having integrated Eq. (27'), we get

$$Q\gamma = \frac{\xi_0(\gamma)}{1 - R/R_0},$$

(28)

in which

$$\xi_0(\gamma) = \frac{3}{4}\,\frac{\gamma}{R_0}\,\frac{dR_0(\gamma)}{d\gamma}\,\frac{(K_0 - L_0/R^2_{0\gamma})^2}{R^2_0[1 - g(R_0)]\,M_0};$$

$$\frac{\gamma}{R_0}\,\frac{dR_0(\gamma)}{d\gamma} = 1 + \frac{R_{\gamma}}{R_0}\,\frac{1}{\gamma(1+k^2) - 1};$$

$$K_0 = R^2_0 - \frac{1}{2} + \left(R_0 + \frac{1}{2}\right)\exp(-2R_0);$$

$$M_0 = 1 - \frac{6}{5}\left(\frac{R_0}{R_{0\gamma}}\right)^2 + \frac{3}{7}\left(\frac{R_0}{R_{0\gamma}}\right)^4;$$

$$L_0 = R^4_0 - \frac{4}{3}R^3_0 + \frac{3}{2}R^2_0 - 2 + (R^3_0 + \frac{5}{2}R^2_0 + 4R_0 + 2)\times$$

$$\times \exp(-2R_0);$$

* Here and from now on the linear dimensions are expressed in neutron free path lengths α^{-1}.

$g(R_o)$ is the transmittance coefficient for a sphere of critical radius as determined by (11).

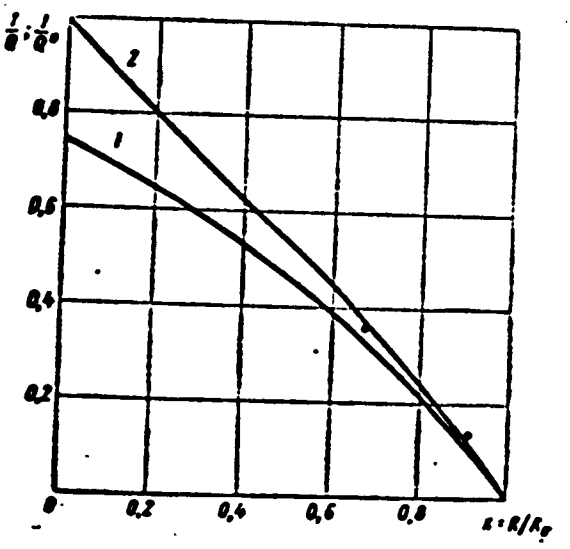

Fig. 2. Multiplication coefficient of a sphere made of $^{Ou}_{(93.5)}$ as a function of the relative sphere size R/R_o : 1) dependence of $1/Q$; 2) dependence of $1/Q^*$; 3) experimental points derived by Stsiborskiy and Kuvshinov; R is radius of the sphere; R_o is the critical radius of the sphere.

Interpolating between the two limiting cases, we get the following approximate expression for $\underline{Q}$

$$Q_\gamma = 1 + \xi_0(\gamma)\frac{R}{R_0 - R}. \tag{29}$$

Fig. 2 shows the dependence of $\underline{Q}(\underline{R}/\underline{R}_0)$ for Ou (93.5), (γ = 0.74; $\xi(\gamma)$ = 0.61) obtained by Eq. (29). It also gives the dependence $\underline{Q}^*(\underline{R}/\underline{R}_0)$, determined by Eq. (20), which can be experimentally checked directly, and we plot the experimental points for $\underline{Q}^*$ obtained by Stsiborskiy and Kuvshinov.

We should point out that the simplified expression (21) in example 2 conveys the correct dependence $\underline{Q}^*(R/R)$ from the qualitatively viewpoint, except that it slightly exaggerates $\underline{Q}^*$, which is an advantage from the point of safety.

Anisotrophy of Angular Distribution

Let us derive a relationship for the mean cosine of angular distribution of neutrons emerging from a sphere. Since $\mu\varphi_0 = 0$, in the critical state, when $\lambda_0 = 1$, the numerator in Eq. (26) remains finite and the mean cosine vanishes on account of the fact that $\underline{Q} \to \infty$. Thus, near the critical state the anisotropy disappears. But it cannot be disregarded completely since the expression for

α_{tr} and β_{tr} contains the product $\overline{\mu Q}$ which remains finite in the critical state. In order to obtain an interpolation expression for μQ, let us continue analytically the dependence of $\overline{\mu Q}$ on R, as determined by (26), to the supercritical state in which $\lambda_1 = 1$. In this limiting case Eq. (26) becomes the following expression

$$\overline{\mu Q} = -\frac{1}{\lambda_1 - 1} \frac{\left(\int \beta F_1(r) \overline{\alpha\chi}(r) r^2 dr\right)^2}{\int \alpha \overline{\chi}(r) r^2 dr \int \beta F_1^2(r) r^2 dr}.$$

in which $F_1(r)$ is the part of the function $\varphi_1[r, (\vec{\Omega}', \vec{\Omega}_0)]$ which does not depend on the angle. Solution of the asymptotic diffusion equation gives us the following expression for φ_1

$$\varphi_{1\,as} = \frac{1}{kr}\left(\cos kr - \frac{\sin kr}{kr}\right)(\vec{\Omega}'\vec{\Omega}_0).$$

Let us assume roughly that

$$F_1(r) = r\left[1 - \left(\frac{r}{R_{1,e}}\right)^2\right],$$

in which

$$R_{1,e} = R_1 + 0{,}71\gamma = \frac{\theta}{k};$$

$\underset{1}{R}$ is the radius of the sphere in the state with $\lambda_1 = 1$;

$$k = \operatorname{tg}(k\gamma);$$

$\theta = 4.4934$ is the root of equation $\Theta = \tan \Theta$.

Integration gives us

$$\bar{p}Q_\gamma = -\frac{\xi_1(\gamma)}{1 - R/R_1},$$

in which

$$\xi_1(\gamma) = \frac{5}{4}\frac{\gamma}{R_1}\frac{dR_1(\gamma)}{d\gamma}\frac{(K_1 - L_1/R_{1\gamma}^2)^3}{R_1^4[1 - g(R_1)]M_1};$$

$$\frac{\gamma}{R_1}\frac{dR_1(\gamma)}{d\gamma} = 1 + \frac{R_{1\gamma}}{R_1}\frac{1}{\gamma(1+k^2)-1};$$

$$K_1 = \frac{2}{3}R_1^3 - R_1^2 + 1 - (R_1^2 + 2R_1 + 1)\exp(-2R_1);$$

(30)

$$M_1 = 1 - \frac{10}{7}\left(\frac{R_1}{R_{1\gamma}}\right)^2 + \frac{5}{9}\left(\frac{R_1}{R_{1\gamma}}\right)^4;$$

$$L_1 = \frac{2}{3}R_1^6 - 2R_1^4 + 4R_1^3 - 5R_1^2 + \frac{15}{2} - \left(R_1^4 + 4R_1^3 \cdots + 10R_1^2 + 15R_1 + \frac{15}{2}\right)\exp(-2R_1).$$

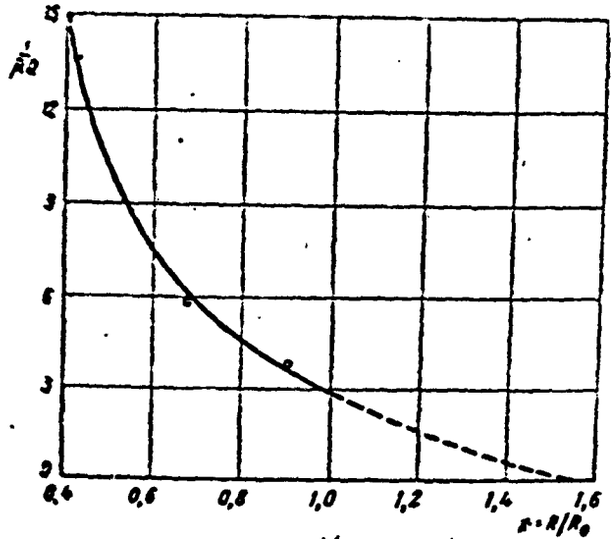

Fig. 3. Dependence μQ on R/Ro for a sphere of Oy (93.5).
Dashed line is extrapolated relationship to the supercritical state with $\lambda = 1$;
o - experimental points of B.D. Stsiborski and M.I. Kuvshinov

The interpolation expression satisfying the limiting cases for Eqs.(27) and (30) takes the form

$$\frac{1}{\mu Q_T} = -\left(\frac{15}{R^2} - \frac{15}{R_1^2} + \frac{1}{\xi_1(\gamma)}\right)\left(1 - \frac{R}{R_1}\right). \tag{31}$$

Fig. 3 shows the dependence of μQ on R/R_0 for Ou(93.5) (R_1 = 3.335; ξ_1 = 0.108) described by Eq. 31, and plots the experimental

points found by Stsiborskiy and Kuvshinov.

Multiple Multiplication by Sphere of Neutrons Reflected From Scattering Medium

Let us determine the number of neutrons emitted by a sphere per primary neutron, taking into account the possibility of their reflection from the layer of scattering medium adjoining the sphere. For the sake of simplicity let us use the concept of the multiplication factor $\underline{Q}^*$ related to $\underline{Q}$ by Eq. (20). Let us single out around the sphere a region of the scattering medium for the given sphere, the reflection coefficient (albedo) of which can be assumed equal to $\underline{A}$. If a neutron strikes the sphere, $\underline{Q}^*$ neutrons leave it after multiplication. Of these $(1 - \underline{A})\underline{Q}^*$ leave the bounds of the separated region, while $\underline{AQ}^*$ neutrons return to the sphere and, after multiplication produce $\underline{AQ}^{*2}$ neutrons of the next generation, and so on. Thus, for each primary neutron the sphere emits $(1-A)Q^*+(1-A)AQ^{*2}+(1-A)A^2Q^{*3}+\ldots=\frac{(1-A)Q^*}{1-AQ^*}$ neutrons. Calculation of the multiple multiplication can be reduced to the replacement of $\underline{Q}^*$ by $(1 - \underline{A})\underline{Q}^*/(1 - \underline{AQ}^*)$ in the critical equations.

To avoid ambiguity in selecting the dimensions of

reflecting layer, we will consider that A is the Albedo of a scattering layer of infiniteness thickness. The error involved in this assumption has a sign which is favorable from the viewpoint of safety. If the radius of the sphere is large or comparable with the neutron free path length in the reflector, in order to find A we can use the asymptotic diffusion theory, in which the expression for the Albedo for an infinite medium surrounding a sphere takes the form /4/

$$A = \frac{1-2D\left(k+\frac{1}{R}\right)}{1+2D\left(k+\frac{1}{R}\right)}, \qquad (32)$$

in which R is the radius of the sphere (in free path lengths in the reflector),

$D = \frac{(\gamma-1)}{\gamma k^2}$ — is the asymptotic diffusion coefficient in the scattering medium (with absorption); the dependence of k on the absorption constant of the medium γ is given by the relationship $k = \text{th}(k\gamma)$.

Let us find an expression for the Albedo of the infinite reflector in another limiting case, when the radius of the sphere

is small compared with the free path length of the neutrons in the reflector medium. Let us assume we are given a neutron density distribution $n(r)$ in the reflector (r is the distance from the center of the sphere). In the element dV in the neighborhood of point $\vec{r}$, $\frac{n\nu}{\gamma} dV$ neutrons are scattered every second. Taking the scattering to be isotropic, we find that $\frac{n\nu}{\gamma} \Phi(rR) dV$ neutrons arrive at the sphere from the elements dV every second, in which $\Phi(r, R) = \int_{\Omega} \exp[-l(\Omega)] \frac{d\Omega}{4\pi}$, $l(\Omega)$ — is the distance between point r and the surface of the sphere in direction $\vec{\Omega}$, Ω, and is the solid angle at which the sphere can be seen from point r. Integrating with respect to dV, we obtain an expression for the flux incident on the sphere, i.e.,

$$J = \frac{4\pi}{\gamma} \int_R^{\infty} \nu n(r) \Phi(r, R) r^2 dr. \qquad (33)$$

The neutron density $n(r)$ can be expanded into two components; $n_{dir}(r)$ is the density of neutrons of direct origin arriving at point $\vec{r}$ from the sphere without colliding with the scatterer nuclei, and $n_{dif}(r)$ the diffusion density. If the radius of the sphere R

is reduced with the neutron stream from sphere J_0 remaining unchanged, the flux of reflective neutrons of direct origin, striking the sphere, is reduced in proportion to R, while the flux component due to scattering of the diffusion neutrons is reduced in proportion to R^2. Hence at low R it can be taken that in (33)

$$vn(r) \approx vn_{np}(r) = \frac{J_0 \Phi(r, R)}{\pi R^2}. \tag{34}$$

Substituting Eq. (34) into Eq. (33), we obtain an expression for the albedo

$$A = \frac{J}{J_0} = \frac{4}{\pi R^2} \int_R^\infty \Phi^2(r, R) r^2 dr. \tag{35}$$

A comparison with the numerical calculation shows that Eq. (35) is equivalent with a high degree of accuracy to the relationship obtained by interpolating Eq. (35) with respect to the limiting cases $R \to 0$ and $R \to \infty$.

$$A_\gamma = \frac{R}{3}\left[\frac{1+\frac{3}{2}R\,\mathrm{Ei}(-2R)}{1+\frac{R}{2(1-\ln 2)}}\right].$$

(36)

Equation 36 expresses the dependence of the albedo of an infinite medium on the radius of the spherical cavity R and the absorption constant γ for a case in which the radius of the cavity is small compared with the length of the free path, and the diffusion approximation is inapplicable. For intermediate R the albedo may be found approximately by interpolating between the dependence curves for A(R) in Eqs. (32) and (36). For γ = 1.1, this interpolation curve is given in example 3. We should point out that as γ increases, the values of R restricting the sphere of application of Eqs. (32) and (36) are greatly increased.

Example 3. Spheres of $^{0}_{u}$u(93.5) are uniformly extended through infinite space in a scattering medium with the absorption constant γ_2 = 1.1 and free path length α_2^{-1} = 10 cm. Let us calculate the dependence of the critical distance between spheres on the sphere radius R by the method set forth in Section 2.

An infinite system is critical when the averaged breeding co-

efficient in the system is equal to unity.

$$\left(Q-\frac{1}{\gamma_2}\right)\frac{pS[1-g(\alpha_1 R)]}{\bar{\alpha}}+\frac{1}{\gamma_2}=1. \tag{37}$$

Let us express the mean probability of extracting neutrons from the beam $\bar{\alpha}$ and density of location of spheres p in terms of the distance between (ku) spheres d

$$\bar{\alpha}=\alpha_2+pS[g(\alpha_2 R)-g(\alpha_1 R)],\quad p=d^{-3},$$

and let us take into account the correction for multiplication of the neutrons. The relationship for the critical distance between spheres d takes the following form

$$d^3=\frac{S}{\alpha_2}\left[\frac{\gamma_2}{\gamma_2-1}\,\frac{Q^*-1}{1-AQ^*}+1-g(\alpha_2 R)\right]. \tag{38}$$

The dependence of $A(R)$ for a medium with $\gamma_2 = 1{,}1$, as determined by Eqs. (32) and (36) is given in Fig. 4. Figure 5 shows the dependence of the critical distance between spheres d on R (curve 1) derived by Eq. (38). For comparison Fig. 5 (curve 2) shows the dependence $d(R)$

obtained by the method of approximate solution of the integral equation put forward in Section 1.

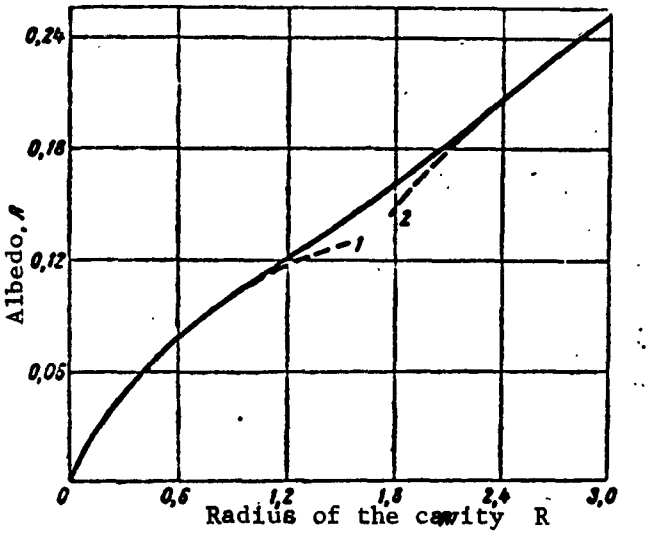

Fig. 4. Interpolated curve of the relationship between the Albedo of the infinite medium and the radius of the spherical cavity R (R expressed in length of free travel of neutrons in the medium, $a_2 = 0.1\ cm^{-1}$, $y_2 = 1.1$).
1 - the curve calculated according to formula (36);
2 - curve calculated according to formula (32).

Figure 5 shows that the results obtained by the integral method contain a greater safety margin than by the method given in Section II. When the radius of the spheres in an infinite system is reduced, both independent curves d(R) merge, since both methods satisfy the limiting transition to a homogeneous mixture.

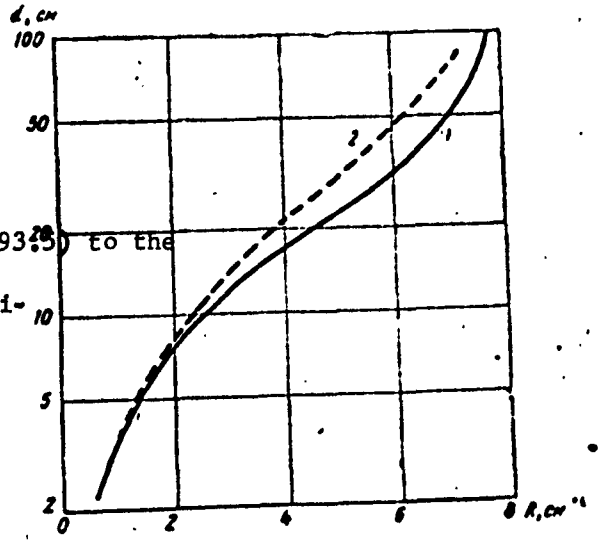

Fig. 5. Relationship of the critical distance between spheres from Oy (93.5) to the radius for a system of spheres uniformally distributed and infinitely scattering medium.

The calculation for a limited system of spheres is not more complicated than in the example considered. If the system has a spherically shaped radius a, for the critical relationship we only need to assume the averaged breeding coefficient (making a correction for multiple multiplication and anisotropy of the angular distribution) to be equal to the critical parameter of the system $1/\gamma(\propto a)$, which is determined by exact methods (dependence of critical radius on optic radius of system given, for example, in /2/.

Limits of Applicability of Methods

Both the methods considered are applicable, irrespective of the relative dimensions of the bodies, distances between them and free path lengths of the neutrons in the media. Although attention has been given in the main to a case in which the scattering medium uniformly fills the space between the multiplying spheres, both methods permit generalization for more complex cases of distribution

of the scattering matter (for example, scattering matter in the form of spherical layers surrounding each sphere in the system, or a system consisting of spheres of two types, and so on). We should recall that by a scattering medium we mean a medium with $\gamma_2 \geqslant 1$, although all the arguments are still valid if medium 2 is a fissile matter with $\gamma_2 < 1$, differing from medium 1 in properties.

The difference in the limits of applicability of the two methods is as follows. Method II is reliable when alculating systems consisting of a large number of multiplying bodies, when the dimensions of the system are large or comparable with the mean probability of extracting neutrons from the beam per unit length during interaction with the spheres, i.e., when we can use the kinetic equation for averaged values. Method I does not have this limitation, since Eq.(6) is valid, irrespective of the number of multiplying bodies in the system.

On the other hand, Method II is applicable to calculation of the system of spheres in a scattering medium surrounded by an external reflector (see example 2), whereas Method I assumes that there are no external reflectors (in principle Method II may be

generalized for this case as well).

We should point out that when deriving critical relationships for systems consisting of a large number of multiplying bodies, it is assumed that the bodies do not form a regular geometric lattice, but on an average are uniformly distributed through the volume of the system, whereas the arrangement of the bodies in actual systems is more often than not geometrically regular. The application of these methods to regular lattices gives a greater safety margin since the presence of singled-out directions in the lattice means a substantial strengthening of the mutual screening of the spheres.

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EXPERIMENTAL STUDY OF INTERACTION BETWEEN TWO SUBSCRITICAL REACTORS

by

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Experiments with Homogeneous Reactors

Experiments were conducted to determine the critical state of a system consisting of two cylindrical homogeneous reactors without reflectors in an ordinary air medium at different distances between them.

The two cylindrical reactors in the experimental plant were attached to a frame, one on a movable base and the other on a carriage moving along the frame. The distance between the cylinders could be varied from 0 to 120 cm.

The walls of the cylinders were made of rustless steel (1kh18n9t) 1.5 mm thick. Each cylinder was fitted with an emergency rod which dropped into the middle of the active zone when the set power level was exceeded. The active zone was a water solution of $UO_2(NO_3)_2$ salt containing uranium with 90% enrichment.

It is obvious that when the reactors interacted, each reactor was subcritical, even when the system was critical as a whole.

The experiments determined the connection between the degree of subcriticality of each reactor (considered separately from each other) and the distance between them at which the reactors form a critical system.

The experiment was carried out in the following way. The cylinders were placed right next to each other and was added at the same time a water solution of $UO_2(NO_3)_2$ until the critical state was reached. The cylinders were then moved quickly apart. The systems became subcritical. In order to achieve criticality, fresh amounts of solution were added simultaneously to both cylinders. The cylinders were separated and filled with fresh amounts of solution until both cylinders became critical, irrespective of each other, i.e., until a position in which the interaction was equal to 0. The inspection and control system was carried out in such a way that it was possible to follow the behavior of each eactor separately, and the system as a whole.

The results of experiments with the homogeneous reactors are shown in Table 1 and in Figs. 1 and 2. In each separate experiment the two interacting reactors were of the same size and had the same active zone composition. The effective breeding factor of the reactor

was calculated by the following formula

$$K_{eff} = \frac{(1 + \varkappa_0^2 \tau)(1 + \varkappa_0^2 L^2)}{(1 + \varkappa^2 \tau)(1 + \varkappa^2 L^2)} \quad (1)$$

in which $\varkappa_0$ is the geometrical parameter of the critical reactor and,

$\varkappa$ is the geometrical parameter of the subcritical reactor.

In experiments 1, 2 and 3 the reactors did not have reflectors, while in experiments 4 and 5 the reactors were set up on a graphite base forming the lower end-reflector and a vertical side wall, along which the mobile cylinder moved. The dependence of $\hat{K}_{eff}$ on the distance between the reactors, found in the first three experiments, is shown in Fig. 2.

Experiments with Heterogeneous Reactors

The experimental plant constituted a steel reservoir, 2.5 meters high, 2.1 meters in diameter, with walls 5 mm thick. At the bottom of the reservoir was a steel base plate with fastenings to which were attached two guiding grids made of (perforated aluminum) for urnaium block channels. The active zone was assembled from uranium blocks with 2 and 10% enrichment. The distance between the reactors

TABLE 1

Results of experiments with homogeneous reactors
(in experiments 1 - 4 D = 300 mm, and in 5 D = 254 mm).

Exp. 1: V_0 = 22,3 л; C = 105 г/л; ρ_H/ρ_5 = 200; k_∞ = 2,032				Exp. 2: V_0 = 29,7 л; C = 74 г/л; ρ_H/ρ_5 = 380; k_∞ = 1,908				Exp. 3: V_0 = 44,95 л; C = 55 г/л; ρ_H/ρ_5 = 520; k_∞ = 1,81				Exp. 4: V_0 = 22,4 л; C = 70 г/л; ρ_H/ρ_5 = 360; k_∞ = 1,9			Exp. 5: C = 70 г/л; V_0 = 35,4 л. ρ_H/ρ_5 = 360		
r, cm	V, l	K_{eff}	H, cm	r, cm	V, l	K_{eff}	H, cm	r, cm	V, l	K_{eff}	H, cm	r, cm	V, l	H, cm	r, cm	V, l	H, cm
0	37,5	0,947	26,5	0	46,6	0,948	33	0	58,0	0,9420	41,0	0	32,8	23,2	0	32,4	32
3	39,4	0,953	27,9	7,5	52,4	0,976	37,4	7,5	68,0	0,9690	48,8	5	35,8	25,3	5	37,1	37
6	40,6	0,973	28,7	15	54,1	0,9820	38,2	15	74,0	0,9780	52,4	15	39,1	27,6	15	44,7	44
12	41,9	0,982	29,6	30	55,9	0,9890	39,5	30	80,0	0,9870	56,6	30	41,6	29,5	30	52,5	52
30	43,46	0,993	30,8	60	57,6	0,9940	40,8	60	85,0	0,9940	60	60	43,4	30,8	60	62,6	62
60	44,2	0,998	31,2	90	58,3	0,996	41,2	90	87,0	0,9960	61,5	110	44,4	31,4	110	70,0	70
90	44,4	0,998	31,3	120	58,4	0,997	41,3	120	87,5	0,9960	62	-	--	—	—	---	--
105	44,6	1,0	31,5	—	—	—	—	--	—	—	—	---	--	---	--	---	--
120	44,6	1,0	31,5	--	--	--	--	--	--	--	--	--	--	--	--	--	--

Note: D is the diameter of the active zone; V_o is the critical volume of one isolated reactor; C is the uranium concentration in the solution; ρ_H/ρ_5 is the ratio of nuclear concentrations of hydrogen and U^{235}; r is the distance between the reactors; V is the critical volume of the two reactor systems; H is the height of the reactor; K_{eff} is the effective coefficient of breeding of one interacting reactor (when considered separately from the other); k_∞ is the breeding coefficient in an infinite medium (for the fifth experiment k_∞ is not calculated).

TABLE 2.

Results of experiments with homogeneous reactors

Exp. 1: R_{equi} = 29,8 cm; P_0 = 24,45 kg, ρ_H/ρ_5 = 287				Exp. 2: R_{equi} = 33,3 cm; P_0 = 15,1 kg, ρ_H/ρ_5 = 287				Exp. 3: R_{equi} 24 cm; P_0 = 8,25 kg; ρ_H/ρ_5 = 578			
r, см	P, кг	$K_{эфф}$	H, см	r, см	P, кг	$K_{эфф}$	H, см	r, см	P, кг	$K_{эфф}$	H, см
0	20,66	0,967	85,3	0,9	23,74	0,980	80,2	4,4	11,2	0,981	92,1
3,4	28,8	0,9826	119	5,4	26,8	0,991	90,6	9,9	13,98	0,994	114,9
7,9	37,9	0,9937	156,5	9,9	28,86	0,997	97,6	15,5	15,2	0,997	125
12,4	42,8	0,9971	176,5	14,4	29,46	0,999	99,6	—	—	—	—
16,9	44,4	0,9980	183,5	18,9	29,7	0,9995	100,4	—	—	—	—
21,4	45,5	0,9985	188	23,4	29,9	1,000	100,8	—	—	—	—
25,9	46,4	0,9989	191,5	—	—	—	—	—	—	—	—
30,4	47,2	0,9994	195	—	—	—	—	—	—	—	
34,9	48	0,9997	198	—	—	—	—	—	—	—	
39,4	48,6	0,9999	201	—	—	—	—	—	—	—	—

Note: R_{equ} (in kg of U^{235}); ρ_H/ρ_5 is the ratio of the nuclear concentrations of hydrogen abd U^{235}; r is the distance between reactors; P is the critical mass of a two-reactor system; (in kg of U^{235}); H is the reactor height; K_{eff} is the effective coefficient of breeding of the reactor calculated from Eq.(1).

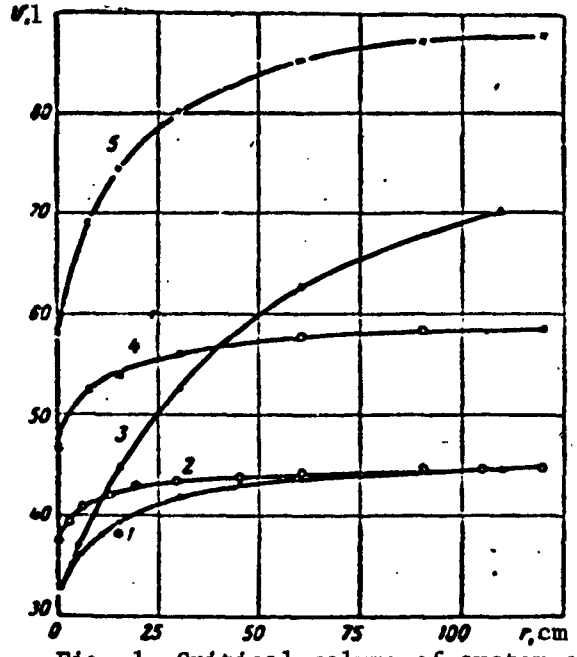

Fig. 1. Critical volume of system of two reactors as function of distance between interacting reactors: 1) fourth experiment; 2) first experiment; 3) fifth experiment; 4) second experiment; 5) third experiment.

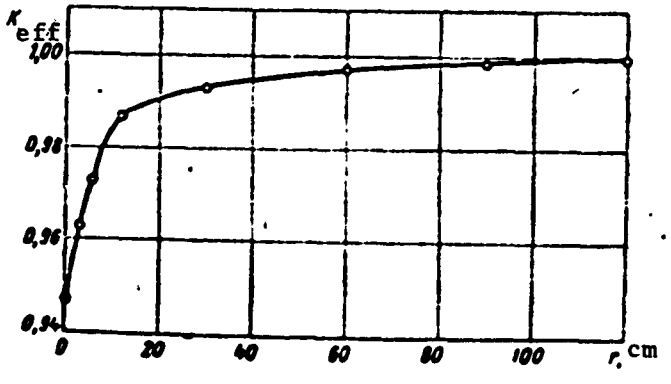

Fig. 2. $\underline{K}_{eff}$ of one of homogeneous interacting reactors as function of distance between them.

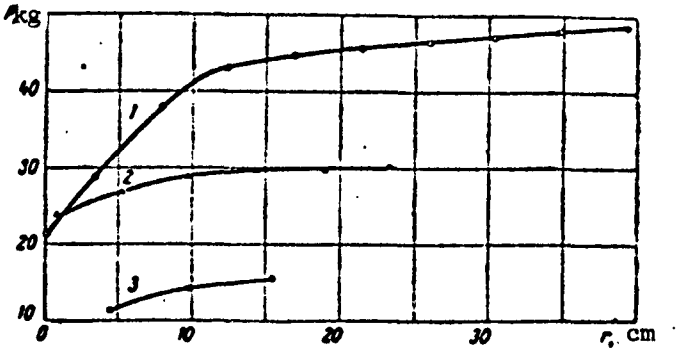

Fig. 3. $\underline{K}_{eff}$ of one of the interacting reactors as function of distance between them: 1) first experiment; 2) second experiment; 3) third experiment.

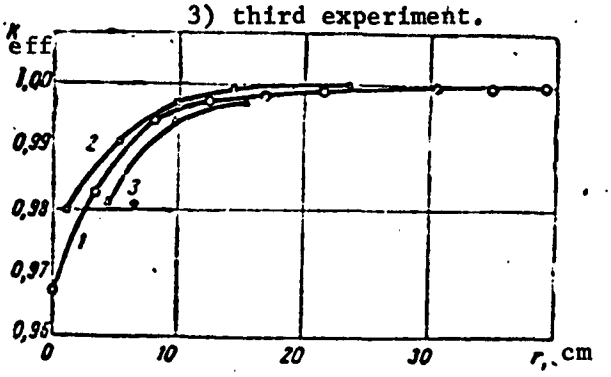

Fig. 4. $\underline{K}_{eff}$ of one of the interacting reactors as function of distance between them: 1) first experiment; 2) second experiment; 3) third experiment.

was changed by reloading the channels in the lattice. The critical state was achieved by filling the reservoir with water. The experimental method was similar to that described for homogeneous reactors. Results of the experiments are shown in Table 2 and in Figs. 3 and 4. The experiments demonstrated that the combination of two roughly equilateral subcritical reactors flush with one another ($\underline{K}_{\underline{eff}}^{\wedge} = 0.94$) is critical for each of them. Two identical subcritical cylindrical reactors with $\underline{N/D} = 1$ and $\underline{K}_{\underline{eff}} < 0.94$ each, not exceeding 2.0 at $\underline{K}_{\underline{\infty}}$, cannot form a critical system.

The effective interaction of two identical subcritical reactors at the given $\underline{K}_{\underline{\infty}}$ (breeding coefficient in an infinite medium) is determined by the solid angle between the adjoining surfaces.

The effectiveness of the interaction of two subcritical reactors in water at large distances is considerably less than in air. If there is a 30 cm protective layer of water between the two reactors, interaction between them can be ignored; here the error in determining $\underline{K}_{\underline{eff}}^{\wedge}$ will not be more than 0.1 %.

CALCULATION OF SECTIONALIZED NUCLEAR POWER PLANTS

by

G. I. Marchuk, B. G. Bubovskiy, V.V. Smelov and Z. N. Milyutina

In connection with the study of sectionalized reactor systems which provide for nuclear energy in subcritical reactors and for a considerable increase in the burn-up depth of the fissile isotope /1/, mathematical calculations of these plants were made and results were obtained which can be applied to certain specific versions of the systems.

The physical system of a sectionalized reactor system considered in this paper is to some extent a further development and generalization of the physical system of the [redacted] PWR reactor /2/.

The mathematical calculations are based on the use of matrix spectrorization of finite-difference reactor equations /3/.

1. We consider a sectionalized reactor system representing a combination of a critical reactor, which is the ignition source for brightening [redacted] neutrons, and subcritical sections arranged in series. On the interface between the critical reactor

and the first subcritical section, and also on the interfaces of the other subcritical sections there are dirt layers which only let the neutrons through in one direction. This system was calculated in cylindrical geometry for one subcritical section.

Fig. 1 is a schematic of a sectionalized reactor system. In the middle there is an ignition critical reactor, 1, then a "black" one for thermal neutrons; the barrier, 2, which is a combination of a layer of uranium 235 of thickness 0.2 cm, a layer of cadmium and a layer of moderator water 2.5 cm thick; subcritical section, 3, adjoins the barrier, and there is a "black" layer of uranium on its outside boundary.

In the uranium of the barrier layer the thermal neutrons from the critical reactor are converted into fast neutrons of the fission spectrum, which pass through the layers of cadmium and water and reach the subcritical section, where they breed. The greater part of the neutrons formed in the subcritical section are unable to pass through the neutron barrier in the reverse direction, since they are moderated by the water and absorbed in the cadmium layer*.

*-When taking into account the effects of the reverse transmission of neutrons by the barrier, as shown by further calculation, the increase in the effective breeding coefficient of a sectionalized system is not greater than 10%.

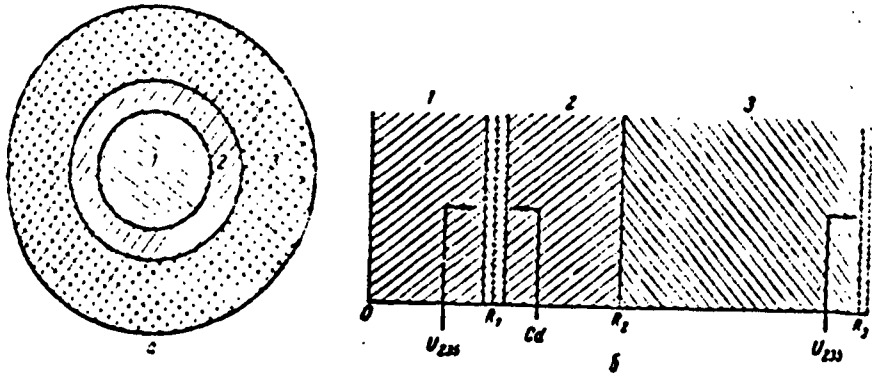

Fig. 1. Sectionalized reactor system: a) top view; b) vertival section; 1) critical reactor; 2)neutron barrier; 3) first subscritical section.

2. The spatial-power spectrum of the neutrons in each subcritical section which is an element of the sectionalized subcritical system can be written in the diffusion-age approximation as the following system of equations /3/

$$\frac{\partial}{\partial u}|\xi\Sigma_s\bar{\varphi}| = \nabla D\nabla\bar{\varphi} - \Sigma_c\bar{\varphi} + \nu f_1(u)\Big(\int_{-\infty}^{u_T}\Sigma_f\bar{\varphi}\,du + \Sigma_{f_T}\bar{\varphi}_T\Big).$$

$$\nabla D_T\nabla\bar{\varphi}_T - \Sigma_{cT}\bar{\varphi}_T = \xi\Sigma_s\bar{\varphi}(\underline{r}, u_T),$$

$$\frac{\bar{\varphi}}{4} - \frac{D}{2}\frac{\partial\bar{\varphi}}{\partial n}\Big|_{S_1} = \eta_1(u), \quad \frac{\bar{\varphi}_T}{4} - \frac{D_T}{2}\frac{d\bar{\varphi}_T}{dn}\Big|_{S_1} = \eta_{1T},$$

$$\frac{\bar{\varphi}}{4} - \frac{D}{2}\frac{\partial\bar{\varphi}}{\partial n}\Big|_{S_2} = 0, \quad \frac{\bar{\varphi}_T}{4} + \frac{D_T}{2}\frac{d\bar{\varphi}_T}{dn}\Big|_{S_2} = 0. \qquad (1)$$

Here ν_f — is the yield of secondary neutrons per 1 productively absorbed neutron,

$\chi(u)$ — is the energy spectrum of neutrons occurring through fission,

$\underline{S}_1$ and $\underline{S}_2$ are the boundaries of the subcritical section with preceding and following sections, respectively,

$\underline{\vec{n}}$ is the normal to the boundary, directed away from the center of the sectionalized system,

$(n)_1$ and n_T are the flux densities /3/ of fast and thermal neutrons flowing from the preceding section to the given one.

If we use the multigroup representation /3/, system (1) reduces to a system of equations of the diffusion type

$$\nabla D_j \nabla \varphi_j - \Sigma_j \varphi_j = -W_j \varphi^{j-1} - S_j \sum_l C_l \varphi_l,$$

$$\frac{\varphi_j}{4} - \frac{D_j}{2}\frac{d\varphi_j}{dn}\Big|_{S_1} = \eta_j, \quad \frac{\varphi_j}{4} + \frac{D_j}{2}\frac{d\varphi_j}{dn}\Big|_{S_2} = 0$$

$$(j = 1, 2, \ldots, m) \qquad (2)$$

If the section under consideration is close to the critical state, the iteration method of solving system (2) is not effective in view of the slow convergence of the solution /3,4/. Hence to solve system (2) we used method /3/ based on matrix representation of the problem (2)

$$\nabla D \nabla \Phi - \Sigma \Phi = 0,$$

$$\frac{\Phi}{4} - \frac{D}{2}\frac{d\Phi}{dn}\Big|_{S_1} = H, \quad \frac{\Phi}{4} + \frac{D}{2}\frac{d\Phi}{dn}\Big|_{S_2} = 0, \qquad (3)$$

in which Φ and H are vectors with the components $\{\varphi_j\}, \{\eta_j\}$, resepctively, $\underline{D}$ and Σ are matrices $D = \| \delta_{jl} D_j \|$, $\Sigma = \| \Sigma_{jl} \|$, and

$$\Sigma_{jl} = S_j C_l + \delta_{jl}\Sigma_j + \delta_{j,\,l+1} W_j,$$

$$\delta_{jl} = \begin{cases} 1, & (j = l), \\ 0, & (j \neq l). \end{cases}$$

In view of the fact that the composition of the barrier is a

thin layer of uranium 235 and cadmium, we should discuss in greater detail the formulation of diffusion conditions on the boundary between the two media (A and B) separated by a thin absorptive layer (C). In the given case it is permissible to consider the problem within the framework of plane geometry. Let us take the direction from medium A to medium B as the positive direction of the normal.

If we use $\varphi(x, E, \mu)$ in the plane problem to designate the neutron flux in a unit of volume of the phase space (x, E, μ), the integral neutron fluxes $\varphi_j(x, \mu)$ of each of the energy groups obviously assume the form

$$\varphi_j(x, \mu) = \int_{E_j}^{E_{j+1}} \varphi(x, E, \mu)\, dE.$$

Here $\mu = \cos\Theta$, Θ is the angle between the normal and the direction of motion of the neutron.

The diffusion approximation corresponds, as is known, to representation of the function $\varphi_j(x, \mu)$ in the form

$$\varphi_j(x,\ \mu) = \frac{1}{2}\left[\varphi_j(x) - 3D\mu\frac{d\varphi_j(x)}{dx}\right],$$

where

$$\varphi_j(x) = \int_{-1}^{1} \varphi_j(x,\ \mu)\, d\mu.$$

Assuming that the neutrons are not scattered in layer C, but ~~xix~~ are only absorbed and reproduced, we can join the number of neutrons entering layer C from medium A every second with the number of neutrons leaving layer C in the direction of medium B every second

$$\varphi_B \int_0^1 \mu d\mu - 3D_B \frac{d\varphi_B}{dx} \int_0^1 \mu^2 d\mu =$$

$$= \varphi_A \int_0^1 \mu e^{-\frac{\Sigma_c d}{\mu}} d\mu - 3D_A \frac{d\varphi_A}{dx} \int_0^1 \mu^2 e^{-\frac{\Sigma_c d}{\mu}} d\mu + Q. \qquad (4)$$

Here $\underline{d}$ is the thickness of the absorbing layer;

$\underline{\Sigma_c}$ is the macroscopic absorbtion cross section in it,

$\underline{Q}$ is the number of neutrons generated per unit of time per $\underline{cm^2}$ of layer.

The balance of neutrons flying in the opposite direction can be written in the form

$$\varphi_A \int_{-1}^{0} \mu d\mu - 3D_A \frac{d\varphi_A}{dx} \int_{-1}^{0} \mu^2 d\mu -$$
$$\varphi_B \int_{-1}^{0} \mu e^{\frac{\Sigma_c d}{\mu}} d\mu - 3D_B \frac{d\varphi_B}{dx} \int_{-1}^{0} \mu^2 e^{\frac{\Sigma_c d}{\mu}} d\mu - Q. \qquad (5)$$

Integrating equalities (4) and (5), we get

$$\frac{\varphi_B}{2} - D_B \frac{\partial \varphi_B}{dx} = E_3(\tau)\varphi_A - 3D_A \frac{d\varphi_A}{dx} E_4(\tau) + Q,$$
$$-\frac{\varphi_A}{2} - D_A \frac{d\varphi_A}{dx} = -E_3(\tau)\varphi_B - 3D_B \frac{d\varphi_B}{dx} E_4(\tau) - Q. \qquad (6)$$

in which $\tau = \Sigma_c d$, $E_n(\tau) = \int_1^{\infty} e^{-\tau t} \frac{dt}{t^n}$.

System (6) easily gives us the boundary conditions for particular cases:

a) Layer C does not absorb neutrons ($\sum_C = 0$);

In this case the system of equations (6) gives the known boundary conditions on the interface between the two media:

$$\left.\begin{aligned} \varphi_A &= \varphi_B \\ D_A \frac{d\varphi_A}{dx} &= D_B \frac{d\varphi_B}{dx} + Q \end{aligned}\right\}; \qquad (6')$$

b) Layer C strongly absorbs neutrons ($\sum_C = \infty$);

In this case we get the boundary conditions on the surface of the "black" layer:

$$\left.\begin{aligned} \frac{1}{2}\varphi_B - D_B \frac{d\varphi_B}{dx} &= Q \\ \frac{1}{2}\varphi_A + D_A \frac{d\varphi_A}{dx} &= Q \end{aligned}\right\}; \qquad (6'')$$

c) Medium B is "black" for neutrons (in particular, it may be a vacuum);

Since it has been assumed above that layer C does not scatter neutrons, a neutron which has left medium A no longer returns; this also means that layer C may be regarded as a "black" one,

thus giving us the condition

$$\frac{\varphi_A}{2} + D_A \frac{d\tau_A}{dx} = Q.$$

6"

In the matrix form the conditions on the boundary between two media separated by a thin absorbing layer can be written in the form of equations (6), in which $\underline{E}_3$ and $\underline{E}_4$ are matrices:

$$E_3 = \| \delta_{jl} E_3(\beta_j) \|, \quad E_4 = \| \delta_{jl} E_4(\beta_j) \|,$$

while $\underline{D}$ is the matrix introduced above.

The differential matrix-vector equation (3) as well as all the boundary conditions were represented in the finite-difference form, and the solution of the problem was found by means of the matrix spectralization method /3/.

To throw light on the physical processes occurring in a sectionalized reactor system, we carried out calculations of this system.

We considered identical, homogeneous lattices in the central critical reactor and in the annular critical section. The uranium-graphite heterogeneous lattice with a 20 by 20 cm mesh consisted

of infinitely long cylindrical rods of uranium, 3.5 cm in diameter, with 2% content of uranium 235.

The concentration of elements in the active zone of the lattice per 1 cm^3 was as follows:

$\rho_{U^{235}} = 0,0^{-}6217.$ $\rho_{U^{238}} = 0.02061.$

$\rho_{Al} = 0,0^{-}4774.$ $\rho_{C} = 0.07977.$

$\rho_{H_2O} = 0,03346.$

The thickness of the uranium 235 layer in the neutron barrier $\underline{\delta}$ = 0.2 cm, and the thickness of the water layer was $\underline{H}$ = 2.5 cm.

Calculations were made for three values of the effective multiplication coefficient ($\underline{K}_{eff}$) in the subcritical section: 0.88, 0.93 and 0.97. In accordance with these values of $\underline{K}_{eff}$, we calculated the dimensions of the section $\Delta\underline{R} = \underline{R}_3 - \underline{R}_2$ (see Fig. 1). The construction of the neutron barrier was the same in all cases.

Figs. 2 - 6 show graphs for the spatial-energy distribution of the neutron fluxes in the cases under consideration.

Figs. 7 and 8 show the fission integral $\underline{Q}(\underline{r})$ for a subcritical

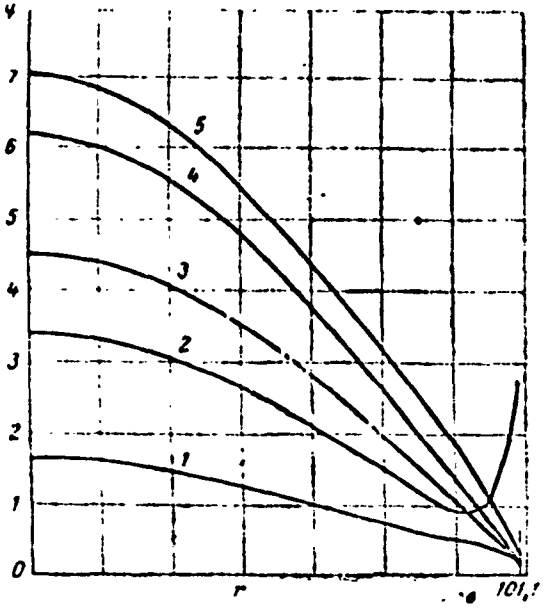

Fig. 2 Neutron flux $\varphi(r, u)$ in critical reactors:

1 — $10^{-1}\varphi$ $(r,\ u = 0{,}75)$; *2* — φ $(r,\ u = 0)$; *3* — φ $(r,\ u = 17{,}5)$; *4* — $10^{-1}\varphi$ $(r,\ u = u_T)$; *5* — φ $(r,\ u = 6{,}5)$.

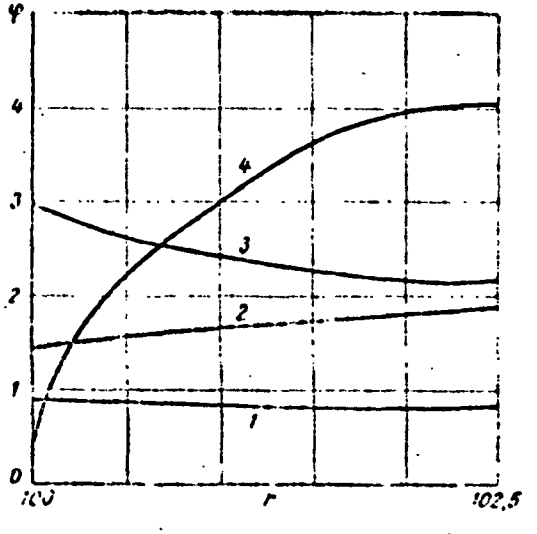

Fig. 3. Neutron flux $\varphi(r, u)$ in water layer (first version):

1 — $10^{-1}\varphi$ $(r,\ u = 0{,}75)$; *2* — φ $(r,\ u = 17{,}5)$; *3* — φ $(r,\ u = 0)$; *4* — $10^{-1}\varphi$ $(r,\ u = u_T)$.

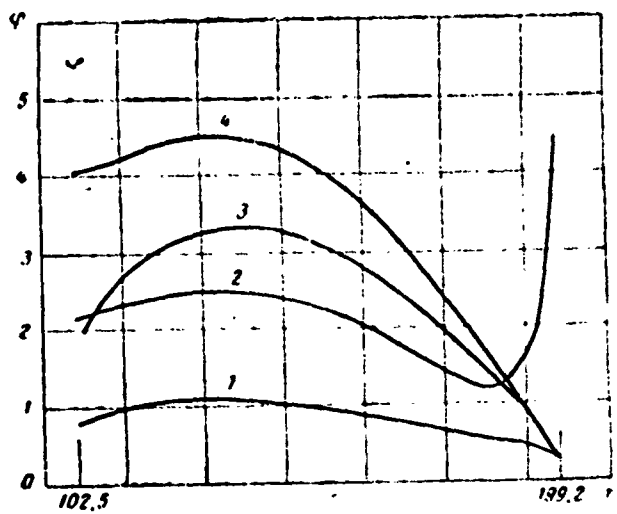

Fig. 4. Neutron flux $\varphi(\underline{r}, \underline{u})$ in active zone of subcritical reactor (first version):
1 – $10^{-1}\varphi(r, u = 0{,}75)$; 2 – $\varphi(r, u = 0)$; 3 – $\varphi(r, u = 17{,}5)$; 4 – $10^{-1}\varphi(r, u = u_T)$.

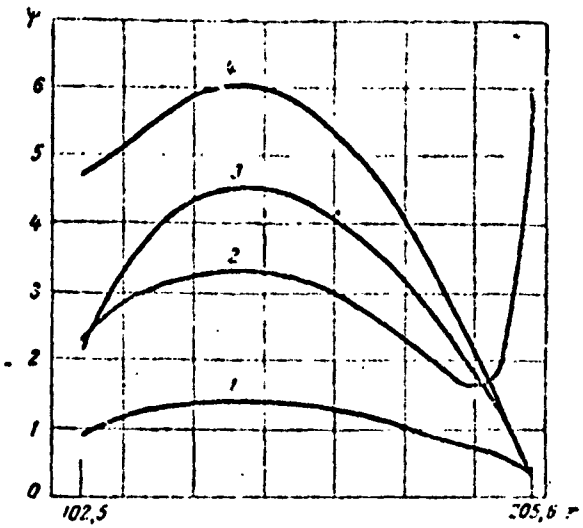

Fig. 5. Neutron flux $\varphi(\underline{r}, \underline{u})$ in the active zone of subcritical reactor (second version):
1 – $10^{-1}\varphi(r, u = 0{,}75)$; 2 – $\varphi(r, u = 0)$; 3 – $\varphi(r, u = 17{,}5)$; 4 – $\varphi(r, u = u_T)$.

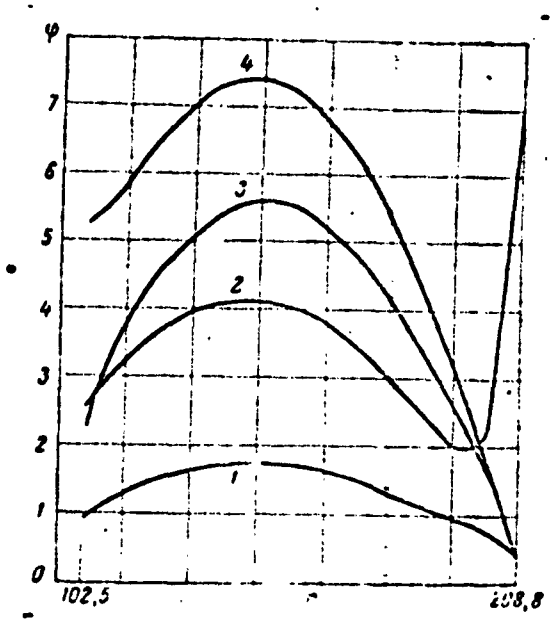

Fig. 6. Neutron flux $\varphi(\underline{r}, \underline{u})$ in active zone of subcritical reactor (third version):

1 — $10^{-1}\varphi$ $(r, u = 0{,}75)$; 2 — φ $(r, u = 0)$;
3 — φ $(r, u = 17{,}5)$; 4 — $10^{-1}\varphi$ $(r, u = u_T)$;

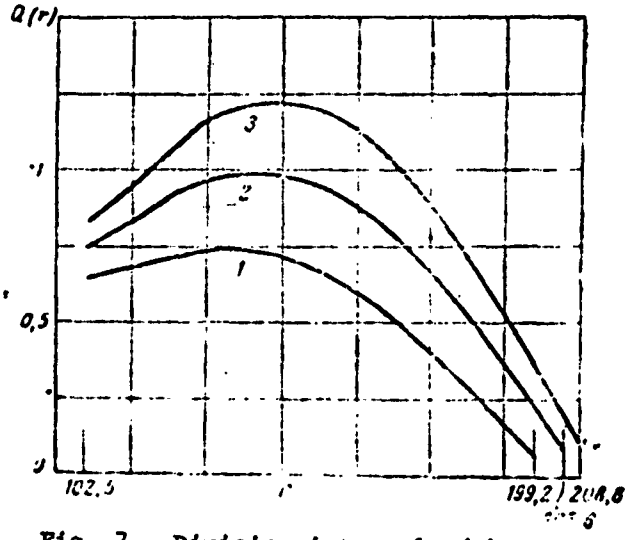

Fig. 7. Division integral $\underline{Q}(\underline{r})$ for a subcritical reactor: 1) first version; 2) second version; 3) third version.

internal reactor.

The fission integral is taken to mean the value

$$Q(\vec{r}) = \nu_f \left| \int_{-\infty}^{u_T} \Sigma_f \varphi \, du + \Sigma_{f_T} \varphi_T \right|.$$

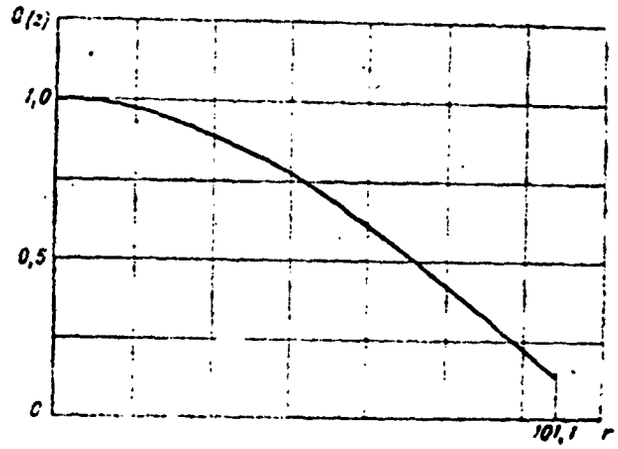

Fig. 8. Division integral $\underline{Q}(\underline{r})$ for critical reactor.

The ratio of the maximum fission integrals $\underline{Q}(\underline{r})$ in the subcritical sections and in the critical internal reactor for different $\underline{K}_{eff}$ of subscritical sections is as follows

K_{eff}^{i}	0,88	0,93	0,97
$\frac{\max \underline{Q}^{i}}{\max \underline{Q}_{cr}}$	0.73	0,99	1.21

As can be seen from the figures given in Figs. 7 and 8, as $\underline{K}_{\hat{eff}}$ in the subcritical reactor increases and approaches unity, there is a considerable increase in the neutron flux in the subcritical reactor through brightening of the high voltage critical reactor by neutrons.

Thus, the calculations given confirm the initial assumptions that the use of sectionalized reactor systems makes it possible to produce high fluxes of thermal neutrons.

/The Ratio of $\underline{Q}_{max}$ to $\overline{\underline{Q}}$ /where $\overline{\underline{Q}}$ is the mean integral value of $\underline{\underline{Q}}(\underline{r})$/ for subcritical and internal critical reactors, and also the ratios of $\overline{\underline{Q}}$ to $\underline{Q}_{\hat{cr}}$ are as follows

	$K^{(1)}_{eff}=0,88$	$K^{(2)}_{eff}=0,93$	$K^{(3)}_{eff}=0,97$	Critical reactor
$Q^{(i)}_{max}/\overline{Q}^{(i)}$ . .	1,40	1,40	1,40	2,12
$\overline{Q}^{(i)}/Q_{cr}$	1,11	1,50	1,84	1,00

The integral values of the energy emission in the critical reactor and subcritical sections as well as the relative values of the specific energy emissions per 1 cell are:

	$K^{(1)}_{eff}$=0.88	$K^{(2)}_{eff}$=0.93	$K^{(3)}_{eff}$=0.97	Critical reactor
. . . . .	3286	4840	6169	1035
Relative specific heat emission per cell	1.09	1.47	1.80	1

It follows from these figures that the distribution of the energy emission ~~from~~ in the subcritical sections may be just as advantageous as in the critical sections, and this confirms the advisability of using the sectionalized reactor system proposed in /1/. It is quite clear that if there were a graphite reactor on the outer boundary of the subcritical section, instead of a layer of uranium, the distribution of the energy release in the subcritical section would be still more advantageous.

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